Making Composition Calculus Great Again

Dominic Zimmer
dependable systems and software
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assisted by Holger Hermanns

Making Com... ulus ... ain

...are

...ersity, Saarbrücken

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Stochastic models

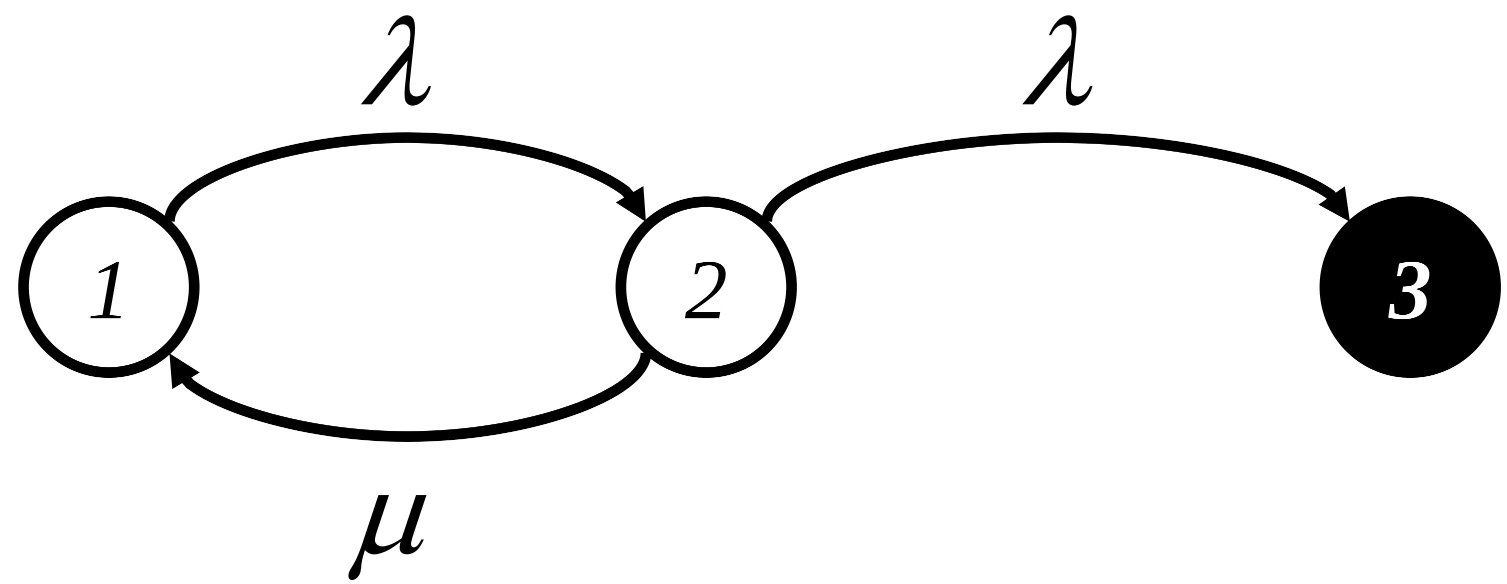
What are they good for?

- Estimate throughput of manufacturing systems.
- Locate bottlenecks in communication systems.
- Assess dependability of satellite systems.
- Calculate performance of cloud infrastructures.
- Plan attacks on cryptoconcurrencies.
- And many other things.

Stochastic models

Their ingredients?

- bubbles
- arrows
- labels
  - of bubbles
  - of arrows



bubbles: ***states***

arrows: ***transitions***

At the pool bar of Mitsis Royal Mare

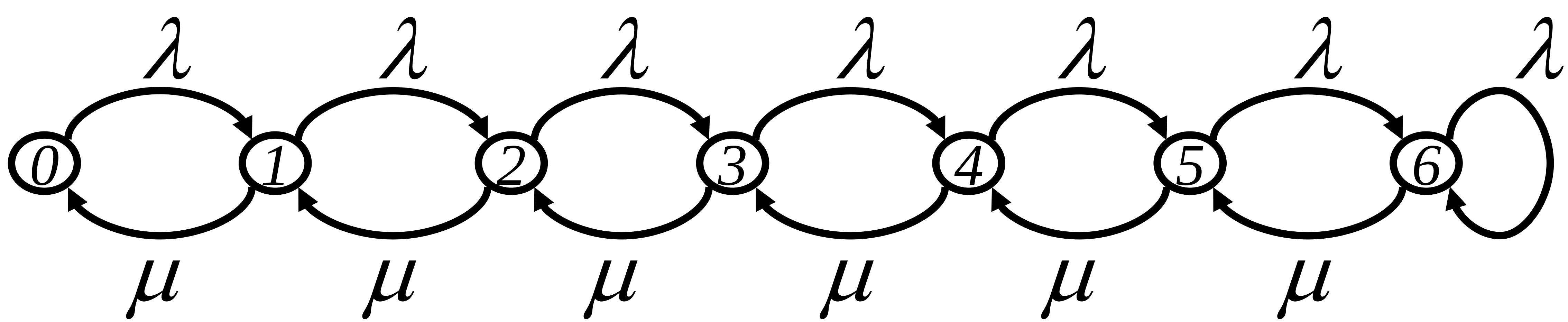
- Customers arrive at a certain frequency,
  say approximately **1** customer per **five** minutes.

  arrival rate λ = **1/5** min

- Service requires, say, **three** minutes.

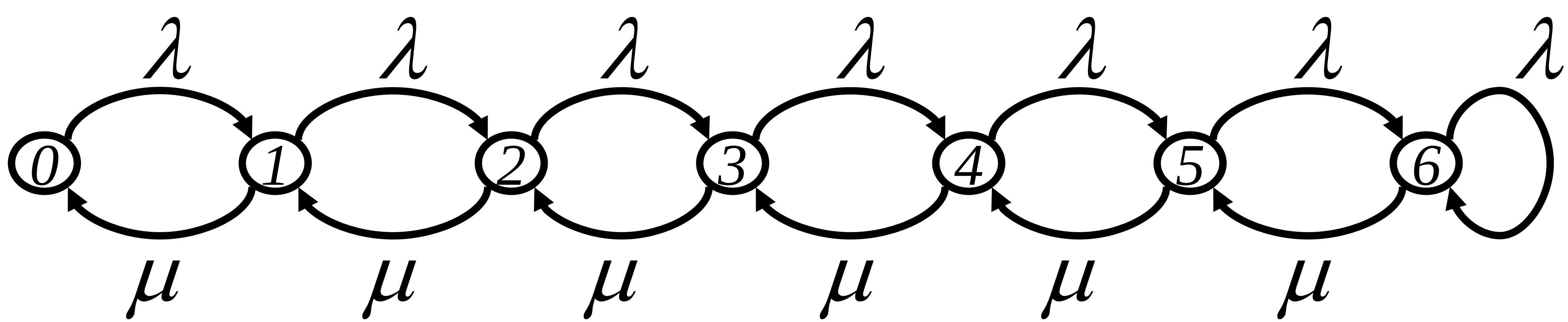
  service rate μ = **1/3** min

- At most **six** customers can wait at the counter.



What is this?

- A ***stochastic process***
- More precise: A ***Markov chain***
- Again more precise: A **finite homogeneous continuous-time Markov chain**



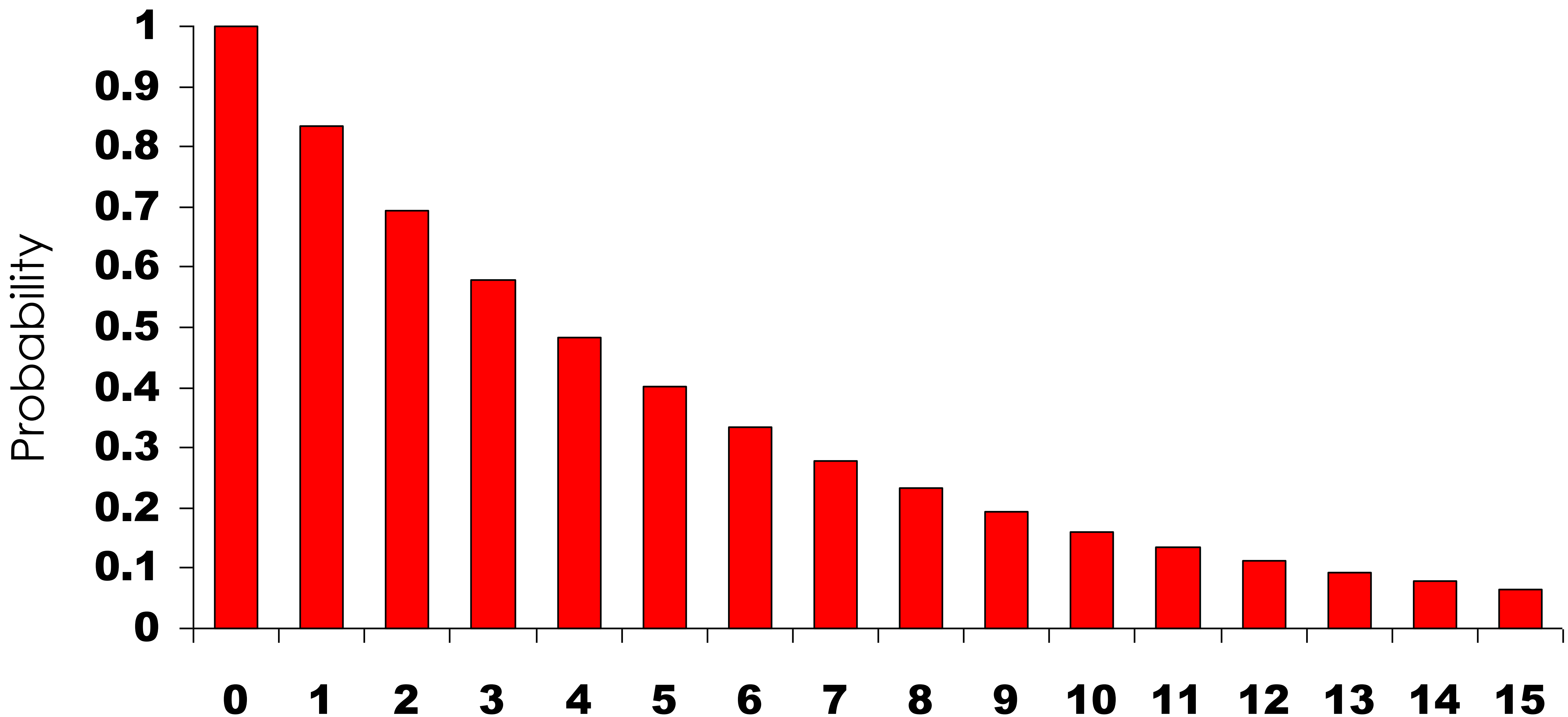
At the door of a gambler

- The gambler rolls a die every minute.

- She comes back once the die shows **6**.
- When will she be back?

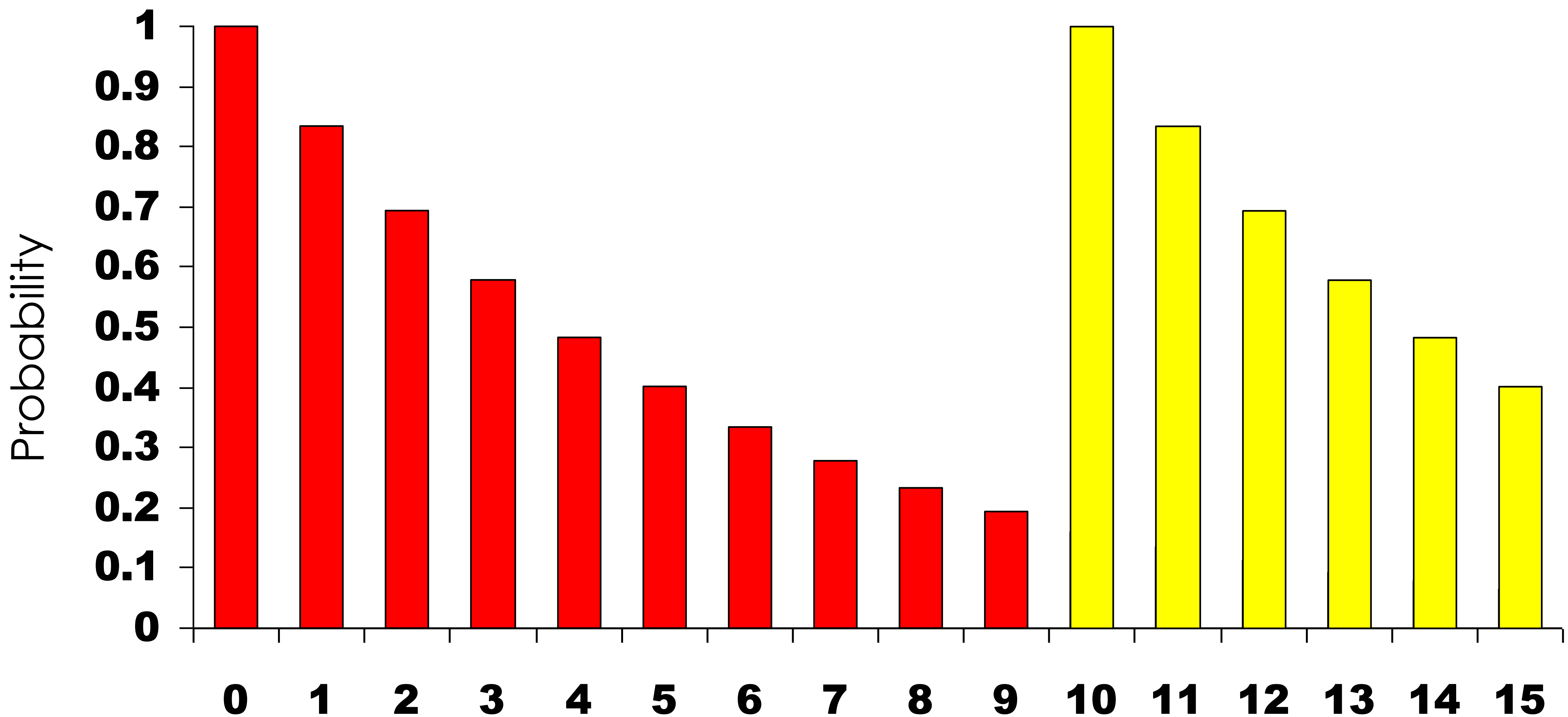
When will she be back, under the assumption that she is not back after 10 minutes?

Is the gambler still absent?



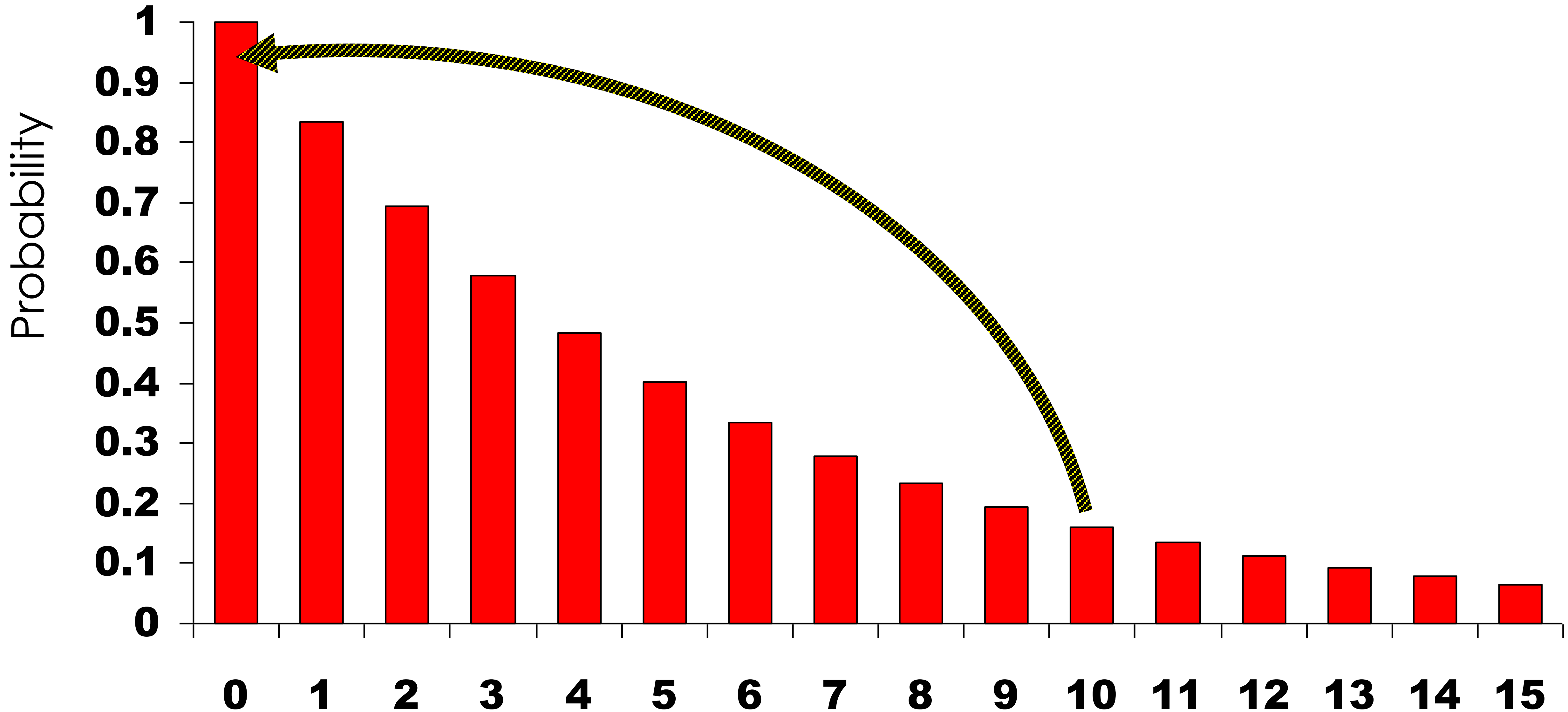
And what if the gambler is still gambling at time ***t* =10** ?

Is the gambler still absent?



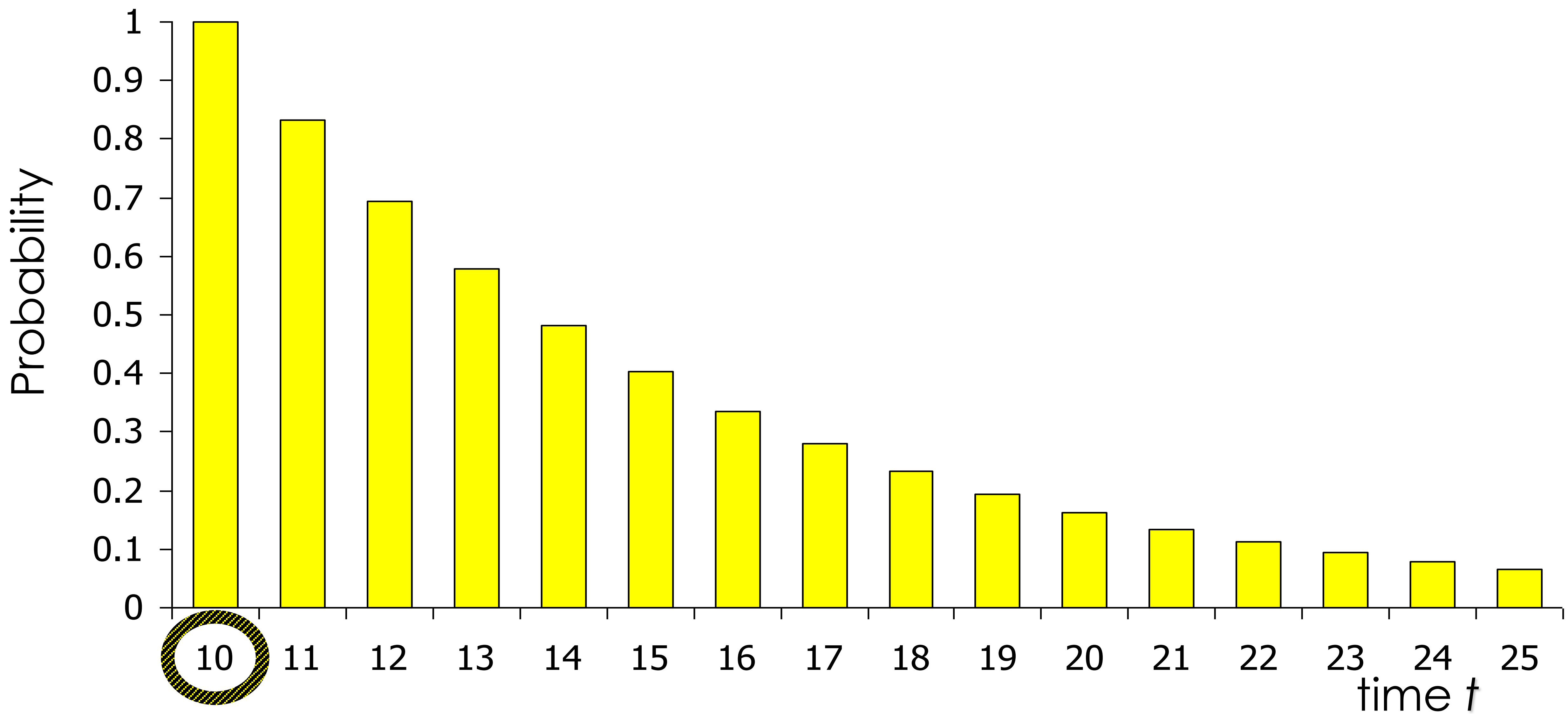
And what if the gambler is still gambling at time ***t* =10** ?

Is the gambler still absent?



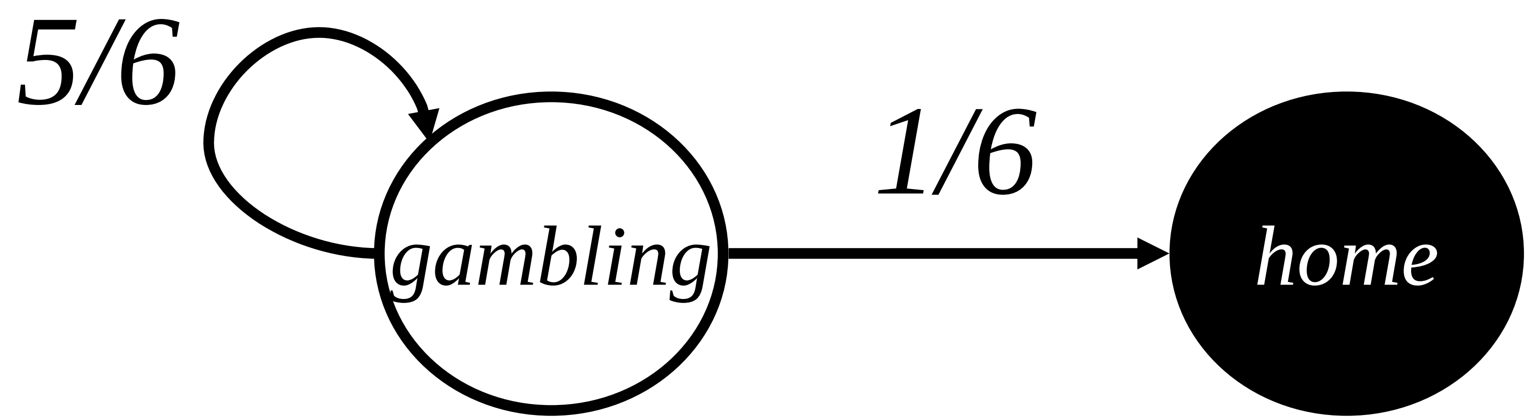
And what if the gambler is still gambling at time ***t* =10** ?

Probabilities remain unchanged



The die has ***no memory!***

Relation to Markov chains

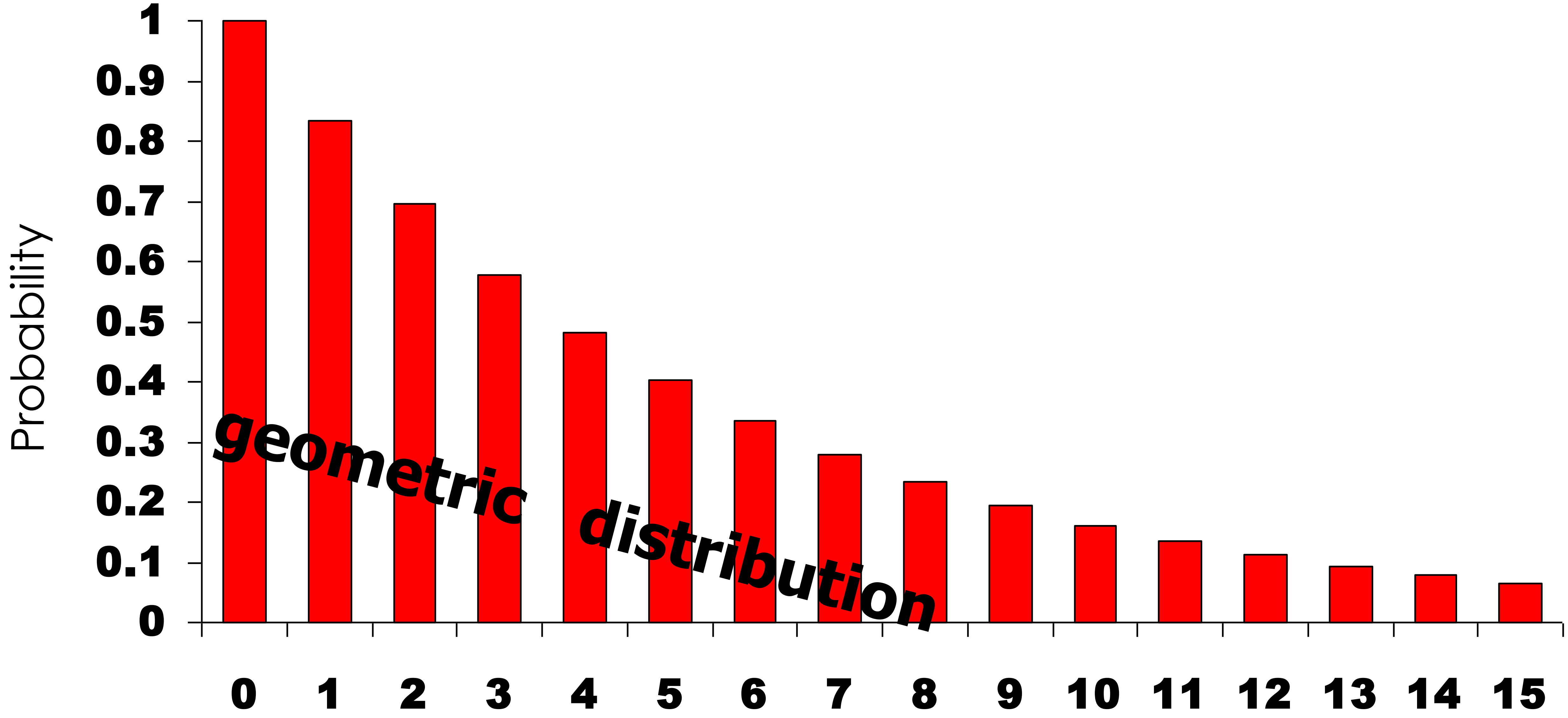


- This ***Markov chain*** describes the gambler's behaviour.

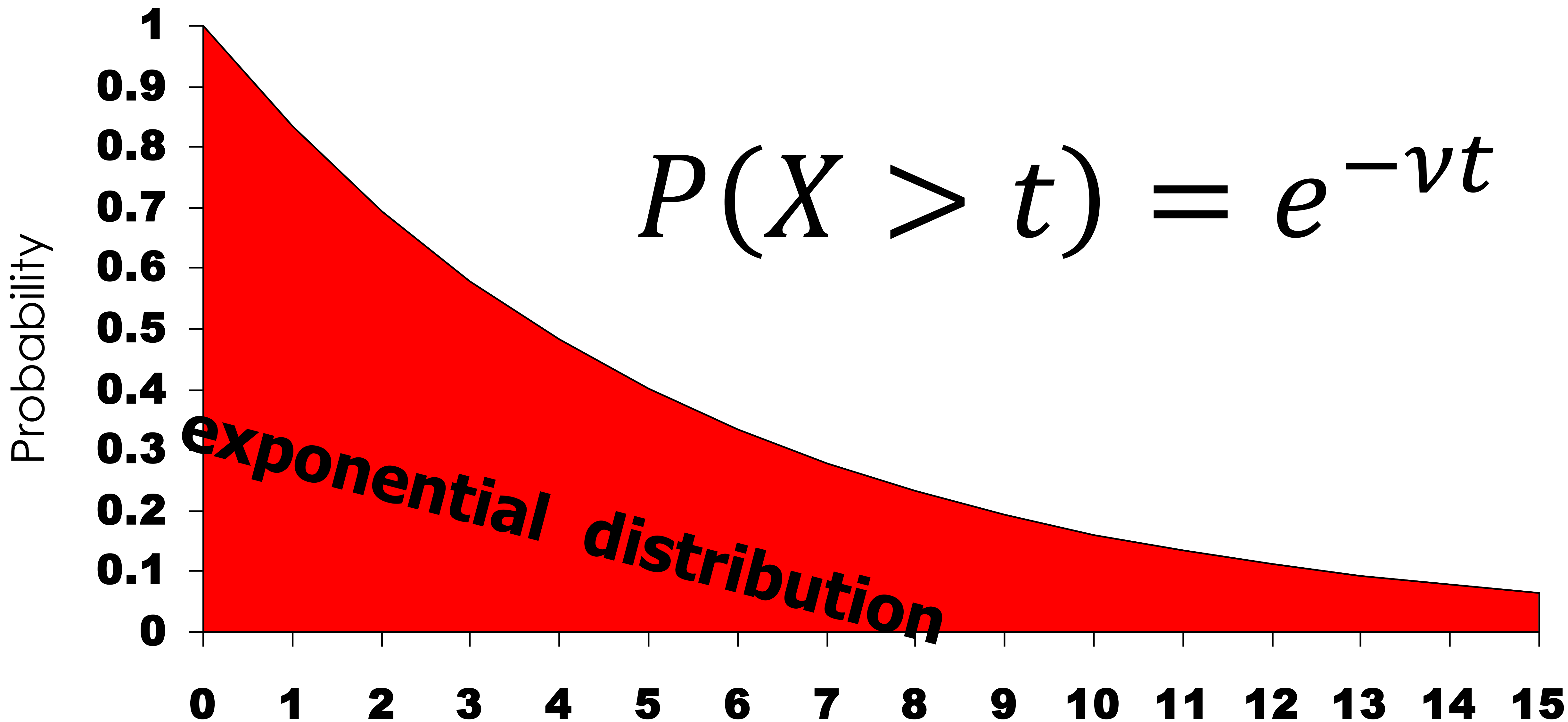
- Markov chains have ***no memory*** of the time spent in their states. They are ***memoryless***.

- Btw:
  The above Markov Chain runs in ***discrete time***.

Discrete time, no memory



Continuous time, no memory

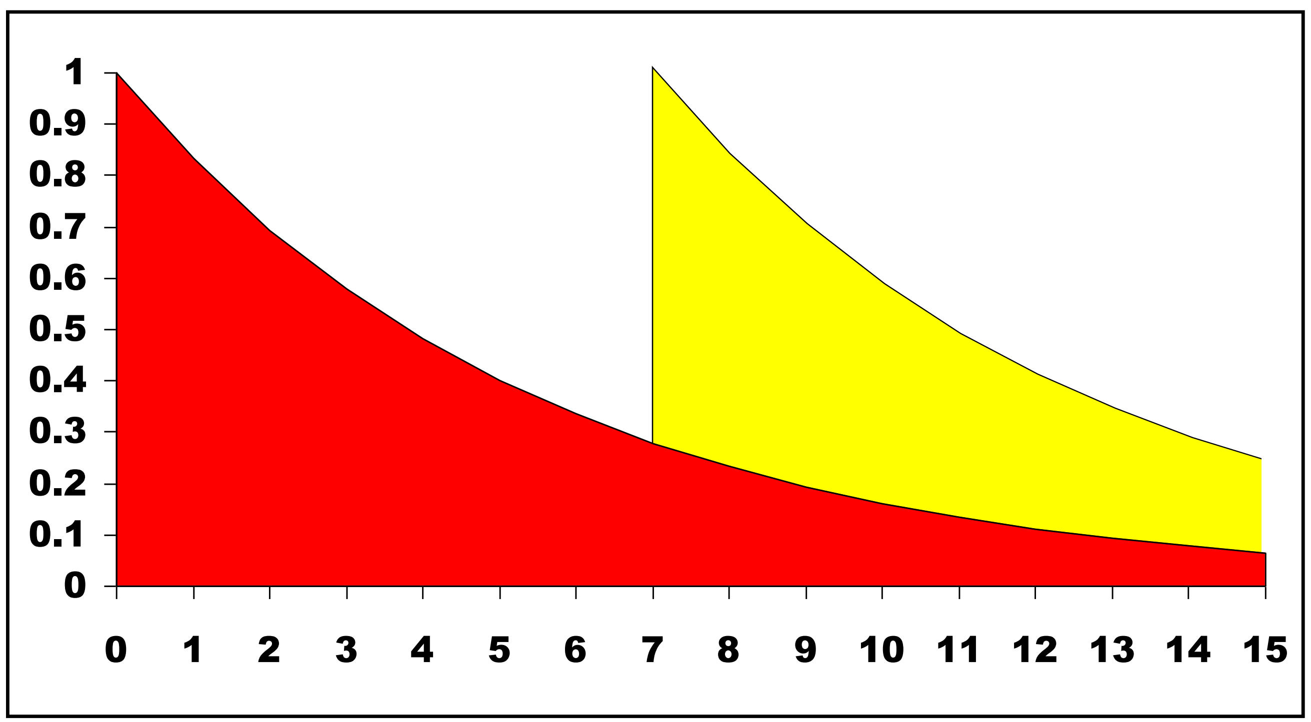


$$P(X > t) = e^{-\nu t}$$

Stochastic models are usually developed in ***continuous time.***

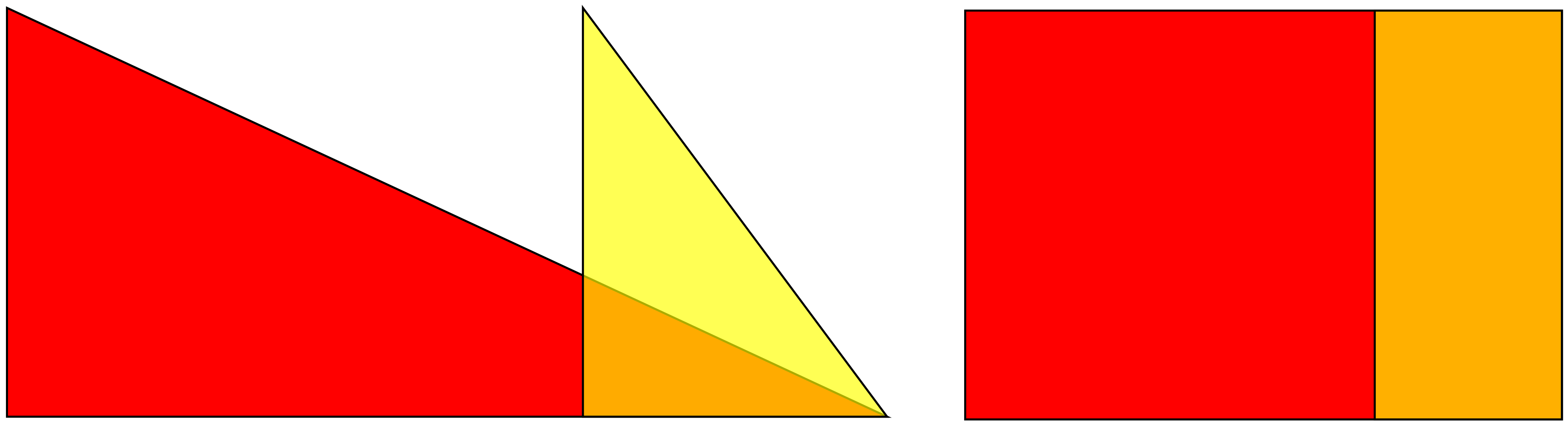
Continuous-time Markov chains

- Automata.
- All durations exponentially distributed.



- Sojourn times in states memoryless.
- Very well investigated class of stochastic processes.
- Widely used out there.
- Best guess, if only mean values are known.
- Efficient and numerically stable algorithms available.

Continuous time, *but* memory

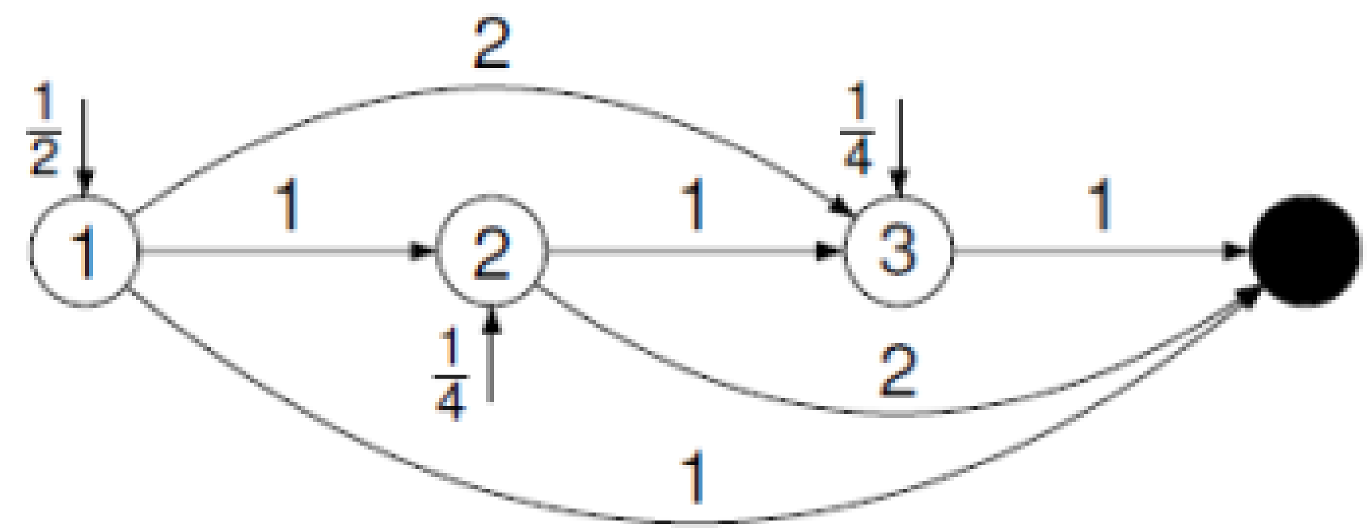


and many, many others.

- Actually: ***Absence of memory is rare, though natural.***
- But: It makes life simpler. (Here: modelling and analysis).
- And: Bitcoin mining is memoryless.
  Stoichiometry is memoryless.
  …

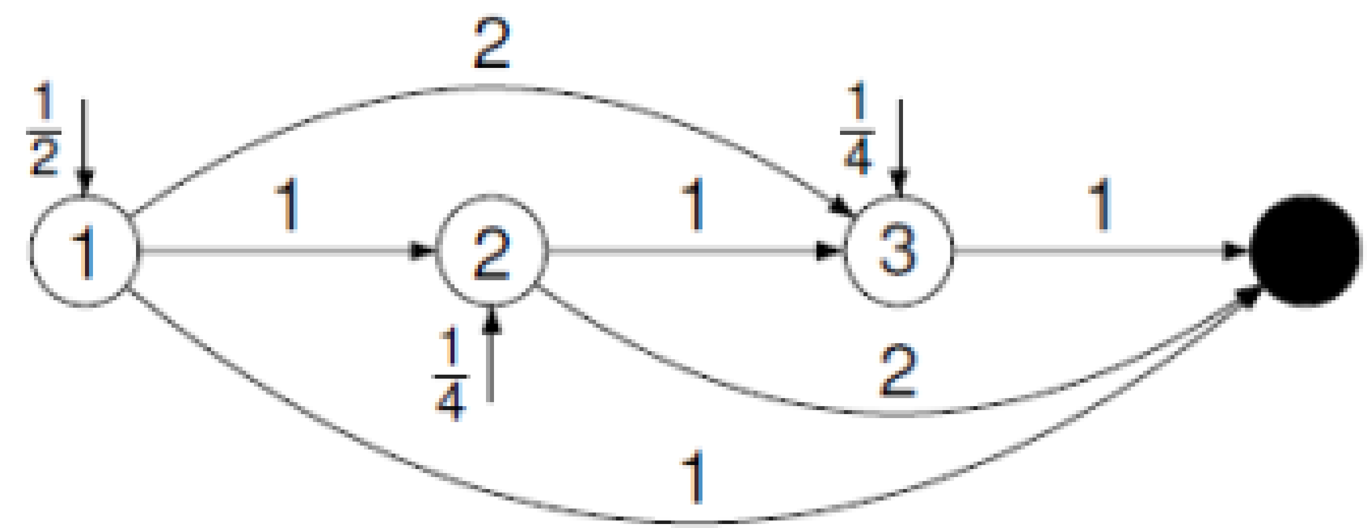
Memoryless Distributions Unleashed

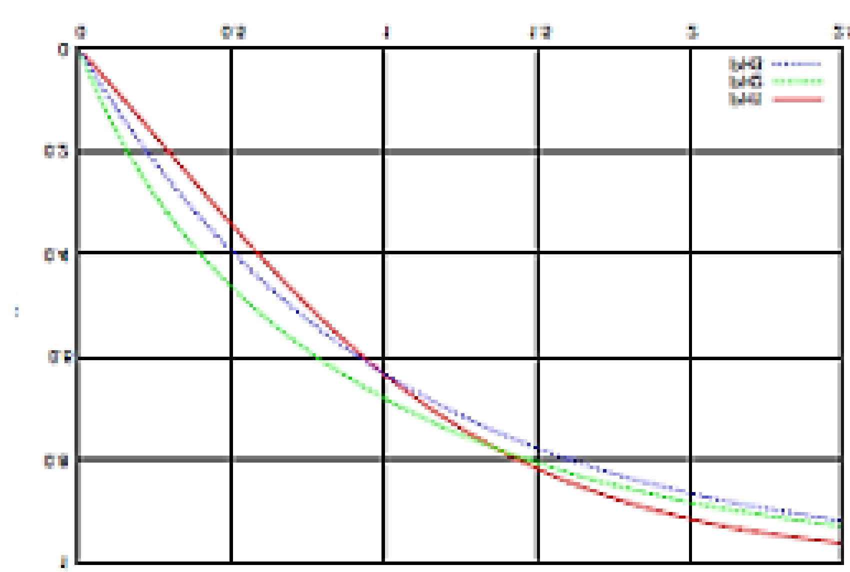
- Absorption time distribution in an acyclic CT Markov Chain.



Acyclic Phase Type Distributions

- Absorption time distribution in an acyclic CT Markov Chain.
- Topologically dense.





Acyclic Phase Type Distributions

- Absorption time distribution in an acyclic CT Markov Chain.

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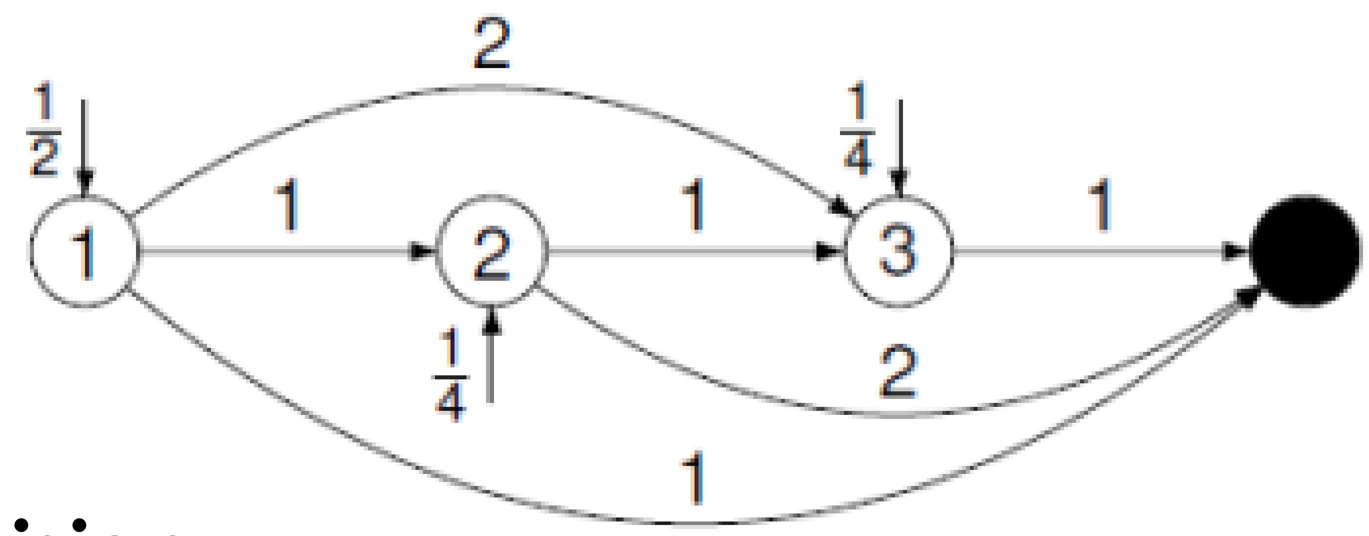
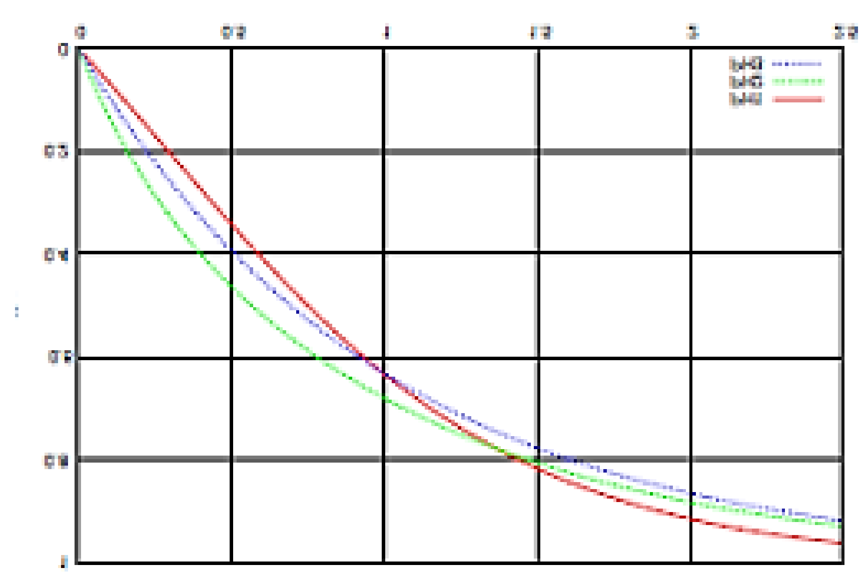


Can approximate arbitrary distributions with arbitrary precision.



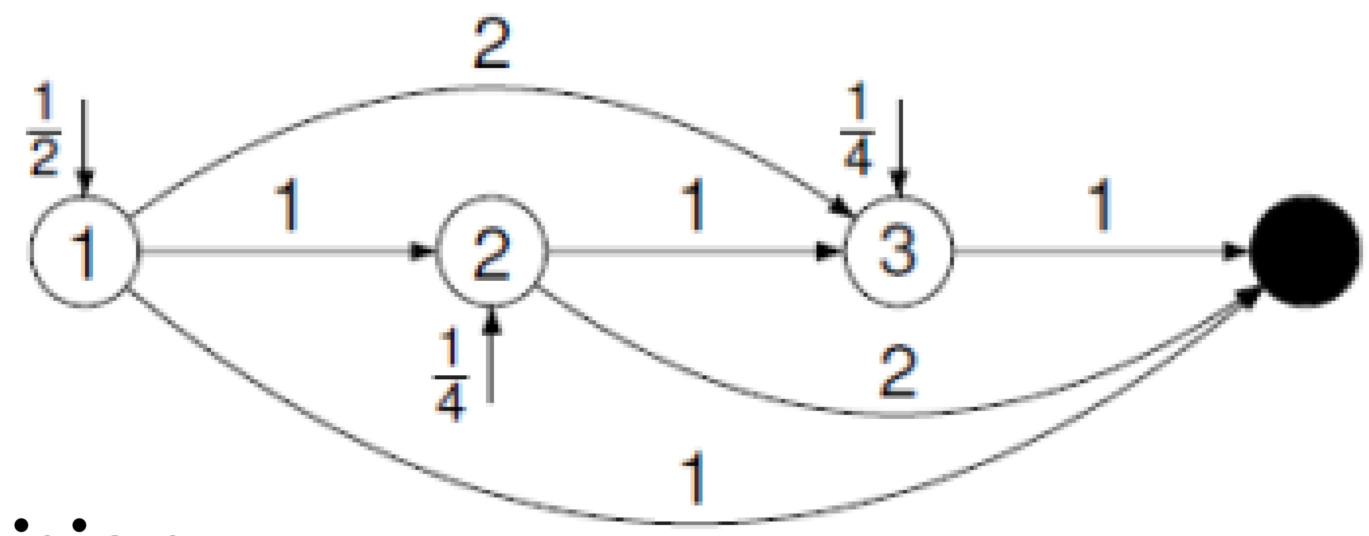
Effective fitting tools are available.

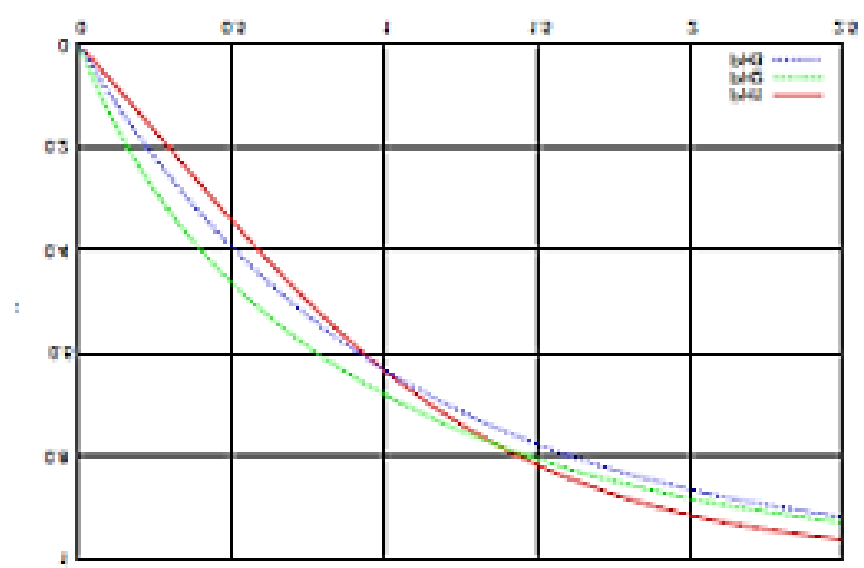
Acyclic Phase Type Distributions – APD

- Absorption time distribution in an acyclic CT Markov Chain.

- Topologically dense:

  Can approximate
  arbitrary distributions
  with arbitrary precision.



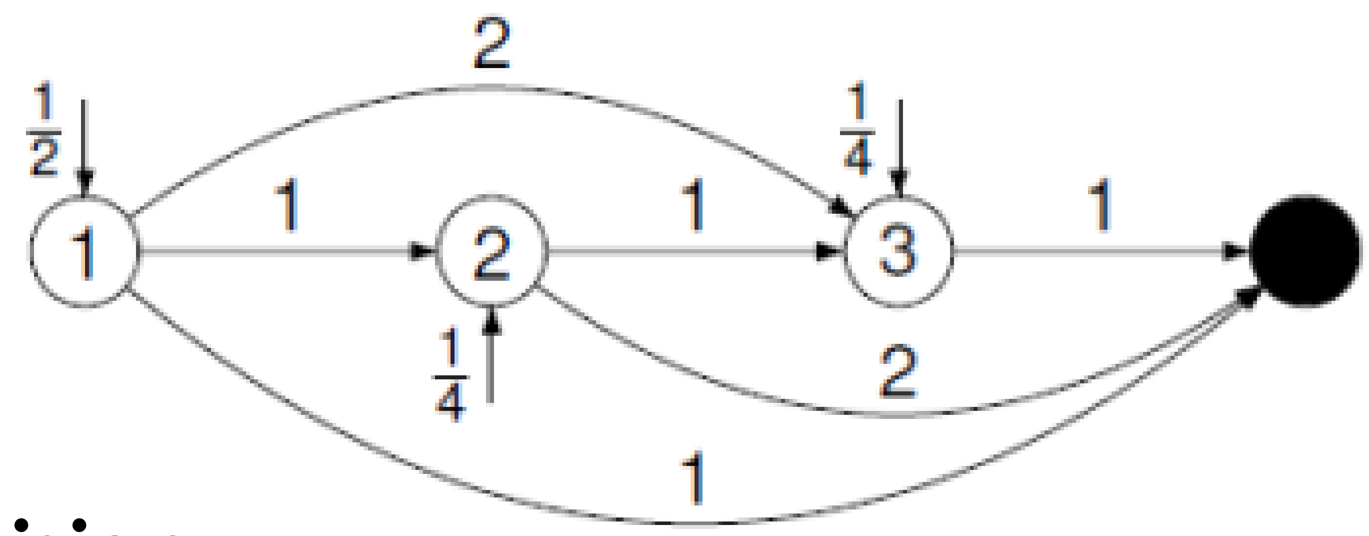


Effective fitting tools
are available.

- Closed under maximum, minimum.

Acyclic Phase Type Distributions – APD

- Absorption time distribution in an acyclic CT Markov Chain.
- Topologically dense:



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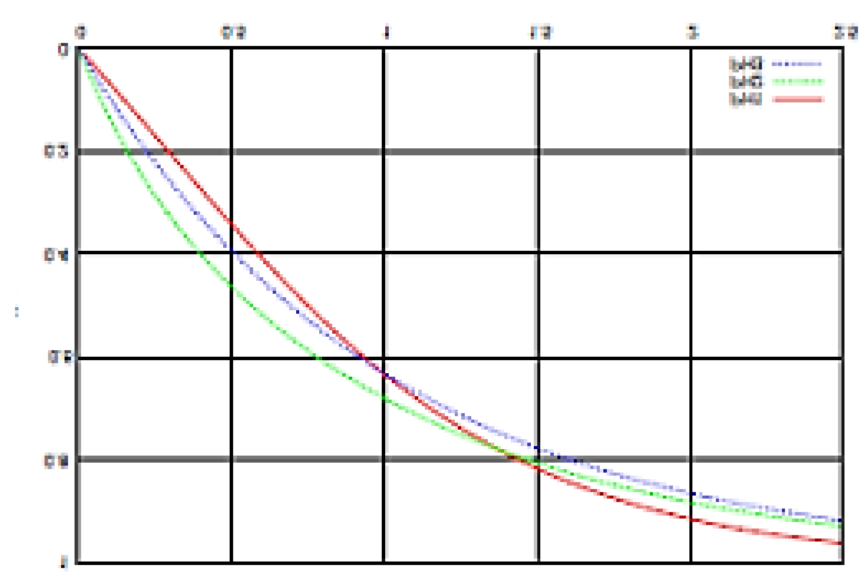
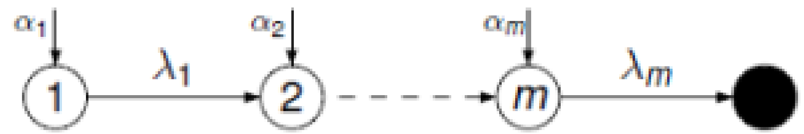


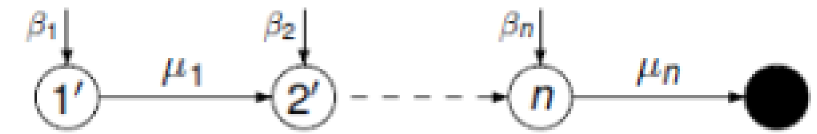
Effective fitting tools are available.

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

- Closed under maximum, minimum, and convolution.

Operations on APD





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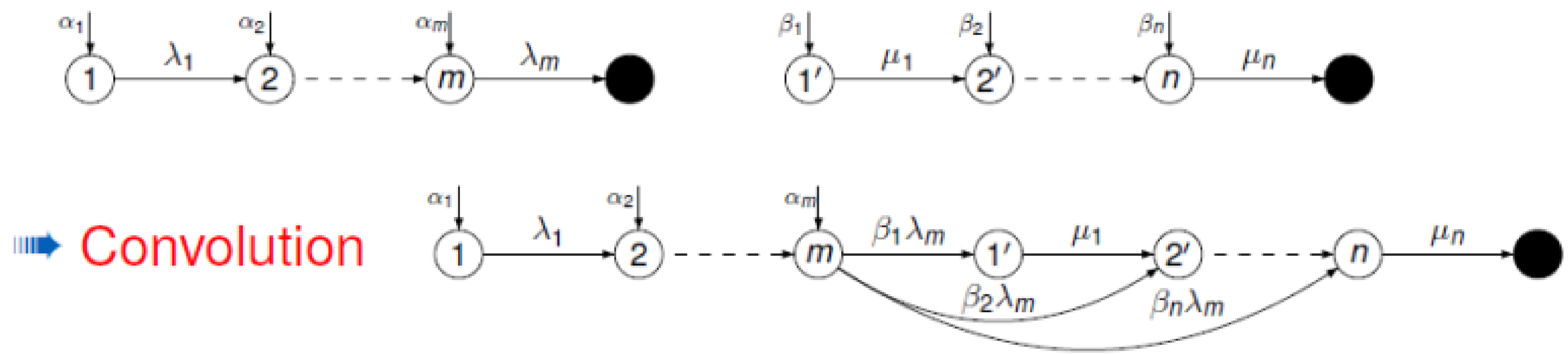

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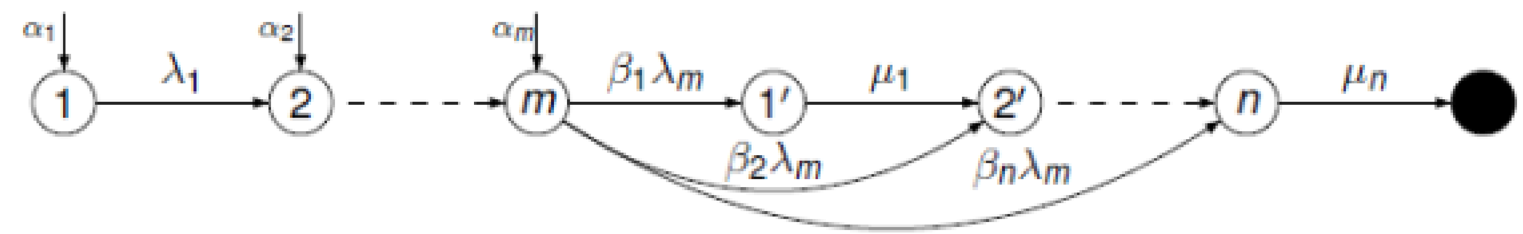
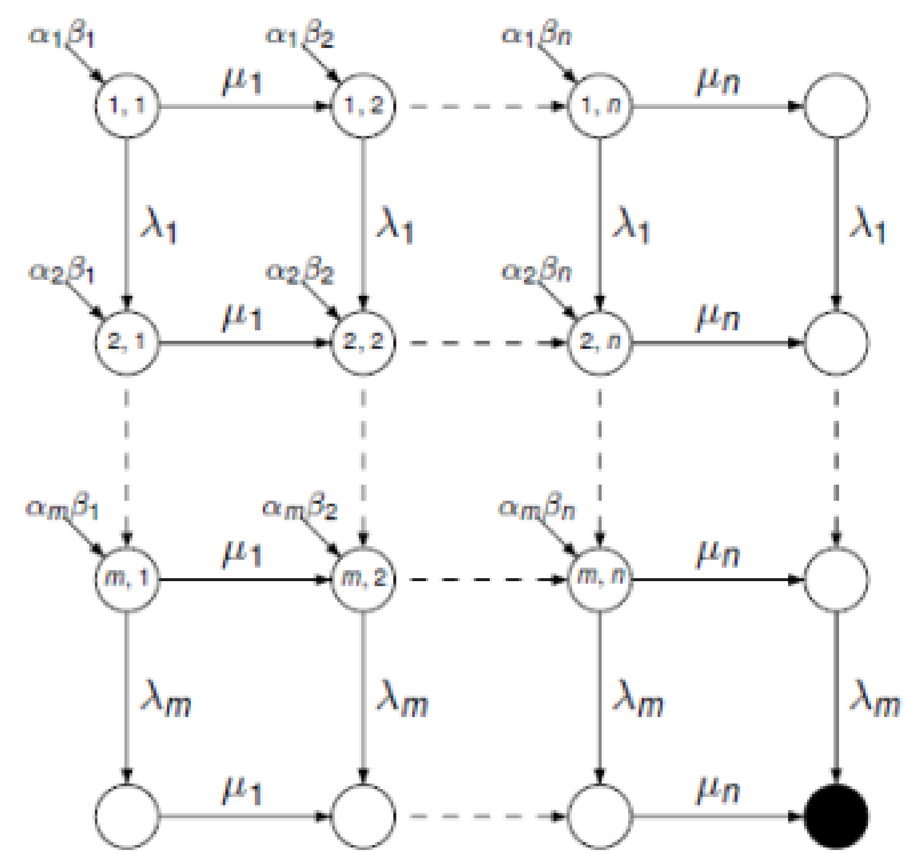
Operations on APD

Convolution



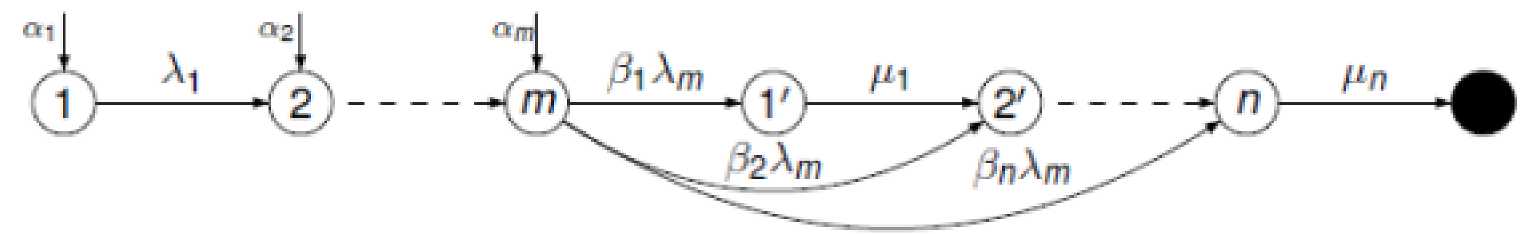
Maximum



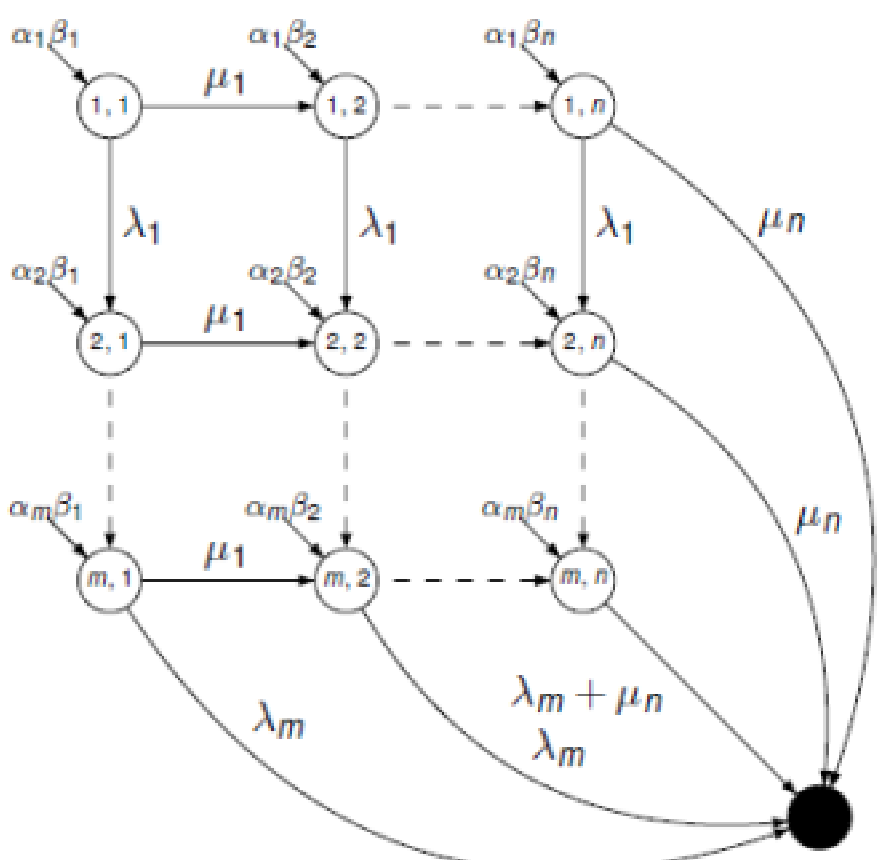
- Closed under maximum, minimum, and convolution.

Operations on APD

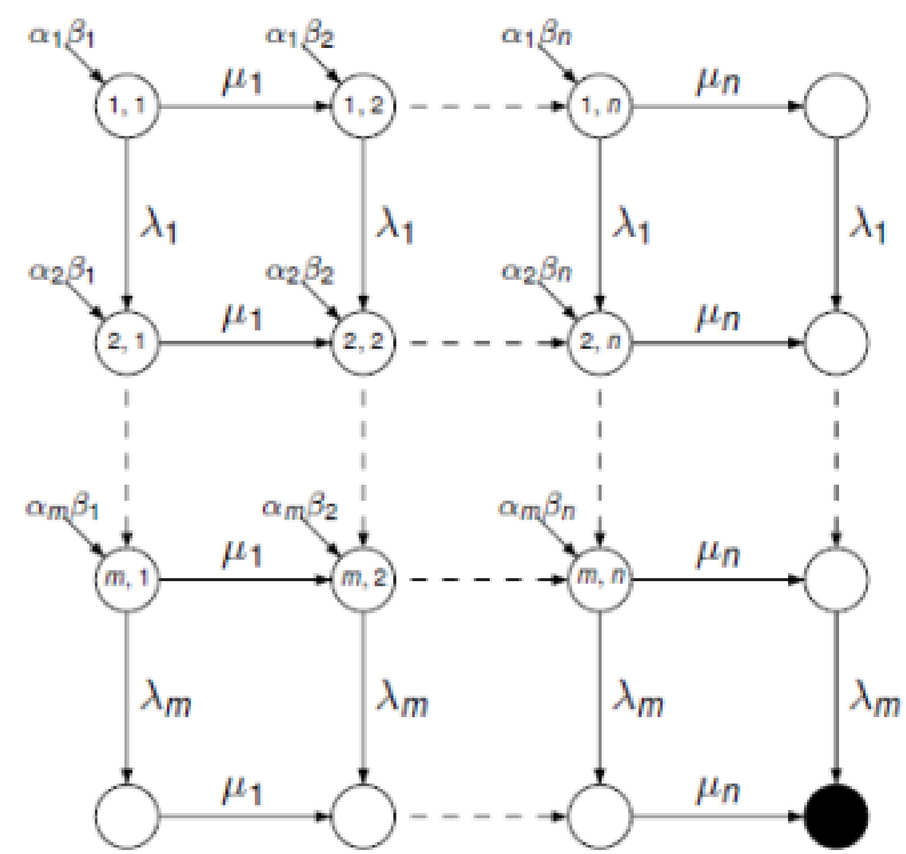
Convolution



Minimum



Maximum



- Closed under maximum, minimum, and convolution.

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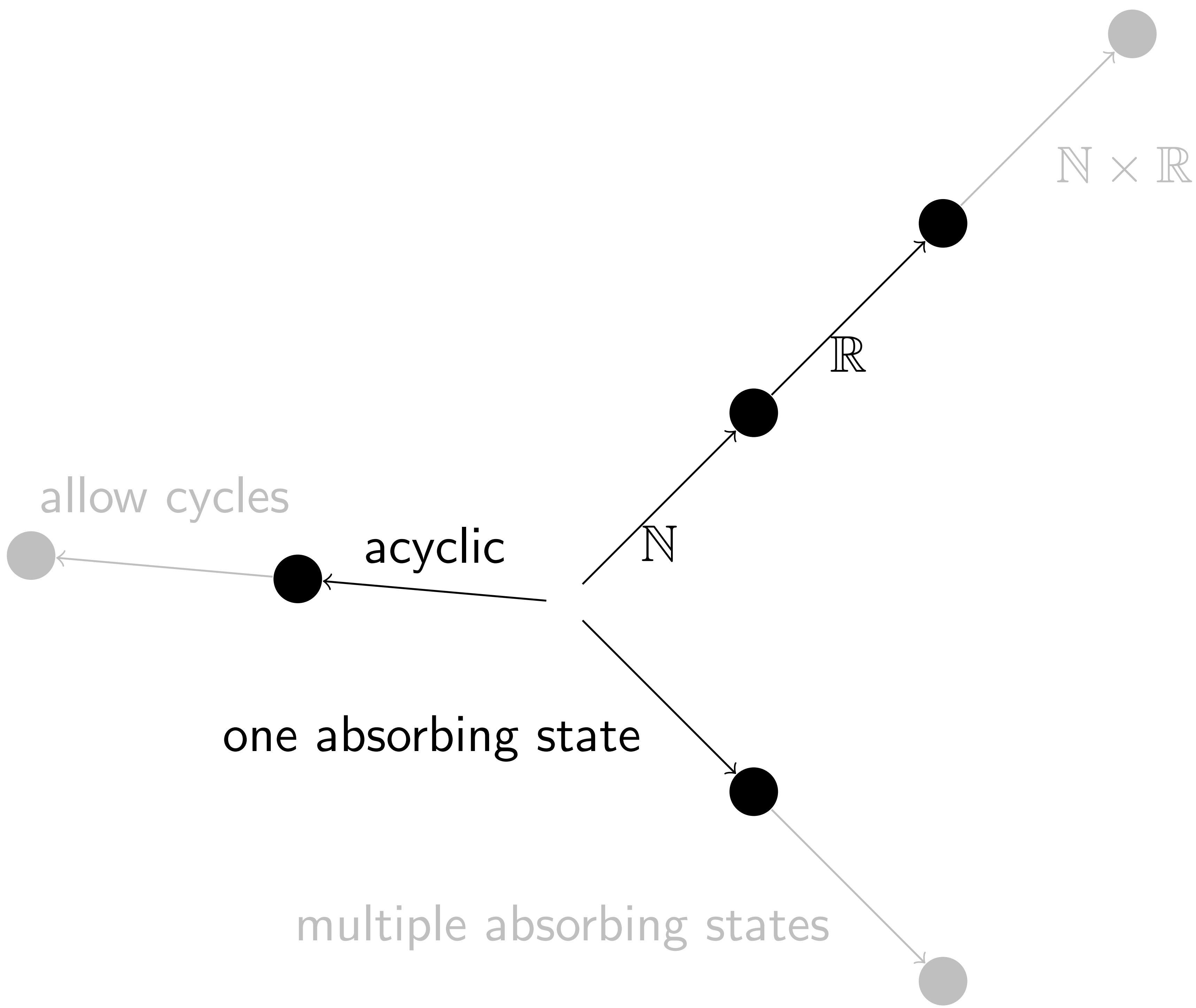
Digging Deeper



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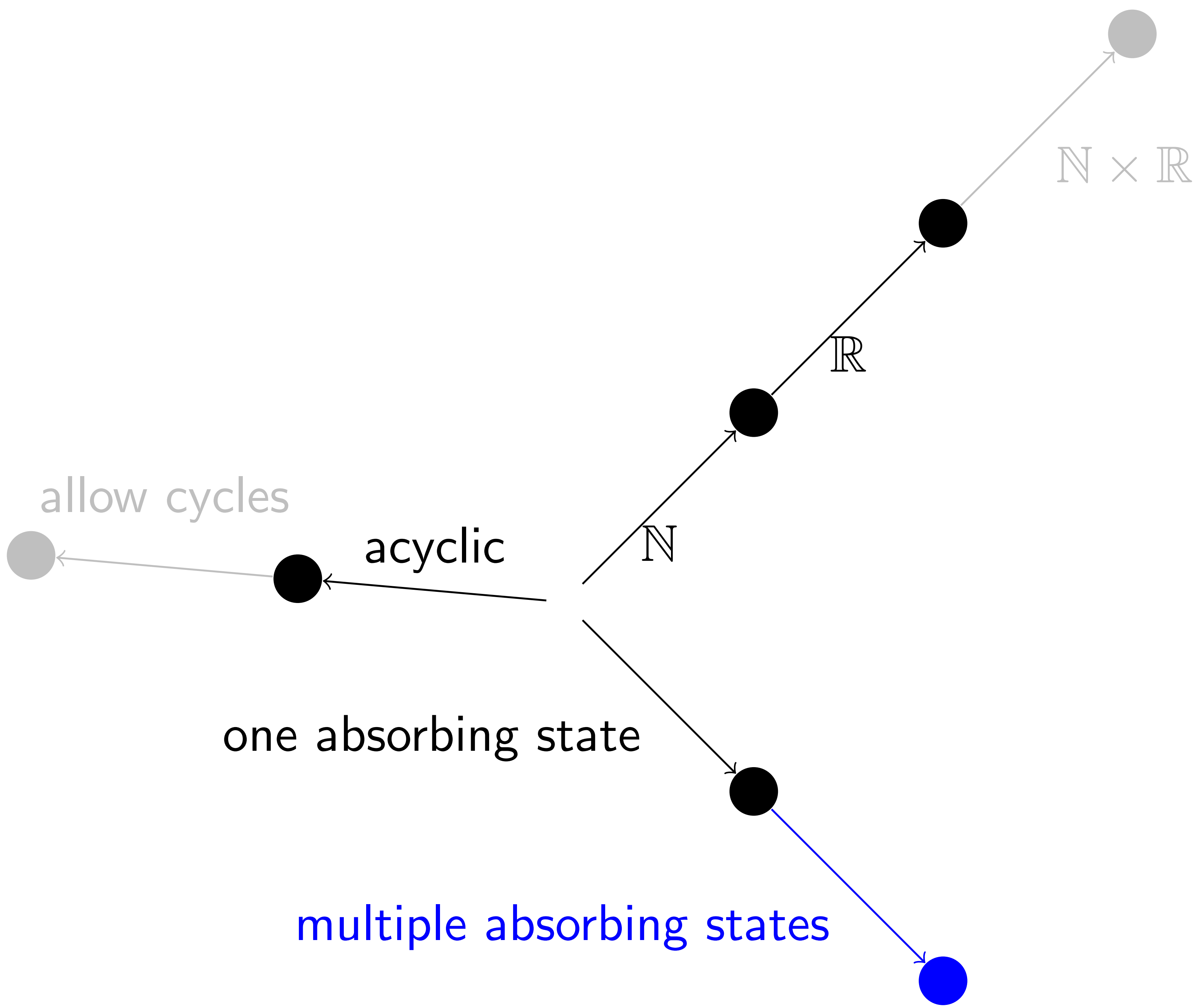
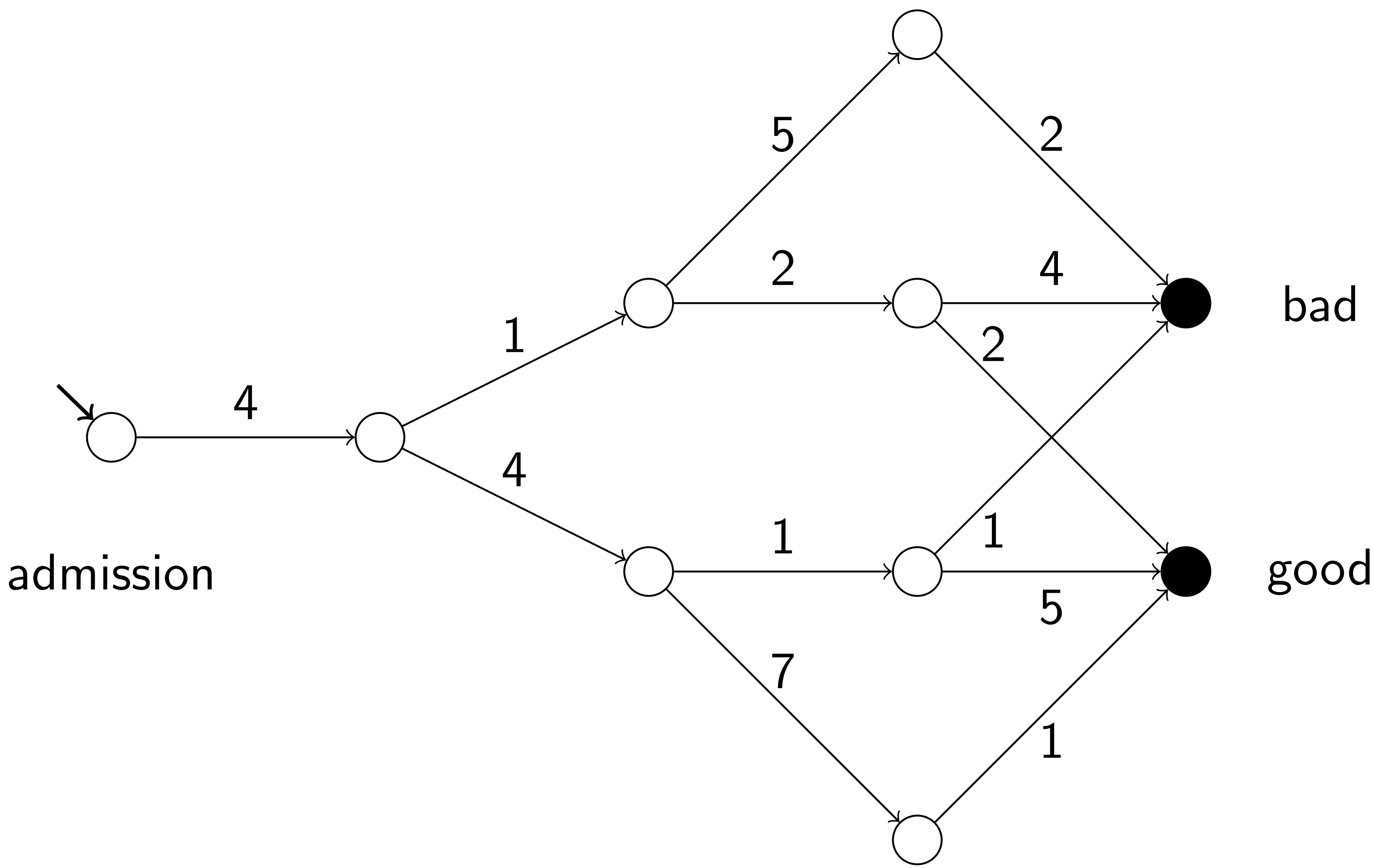
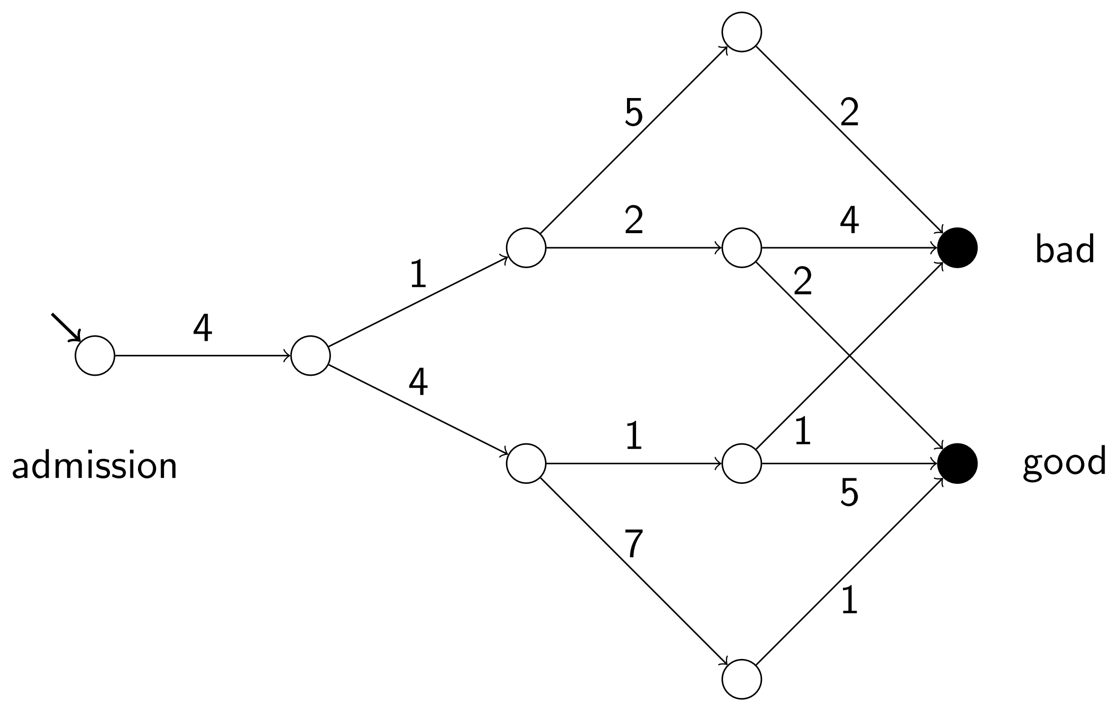


There are two ways out of the Hospital



There are two ways out of the Hospital [2, 1]



$$f_{bad,1} = \frac{1}{5} \cdot \frac{5}{7} \cdot \text{Exp}(4) * \cdots * \text{Exp}(2)$$

$$f_{bad,2} = \frac{1}{5} \cdot \frac{2}{7} \cdot \frac{4}{6} \cdot \text{Exp}(4) * \cdots * \text{Exp}(6)$$

$$f_{bad,3} = \frac{4}{5} \cdot \frac{1}{8} \cdot \frac{2}{7} \cdot \text{Exp}(4) * \cdots * \text{Exp}(7)$$

$$f_{bad} = f_{bad,1} + f_{bad,2} + f_{bad,3}$$

System described by $f = \begin{pmatrix} f_{bad} & f_{good} \end{pmatrix}$.

Multi-Exit Acyclic Phase-Type

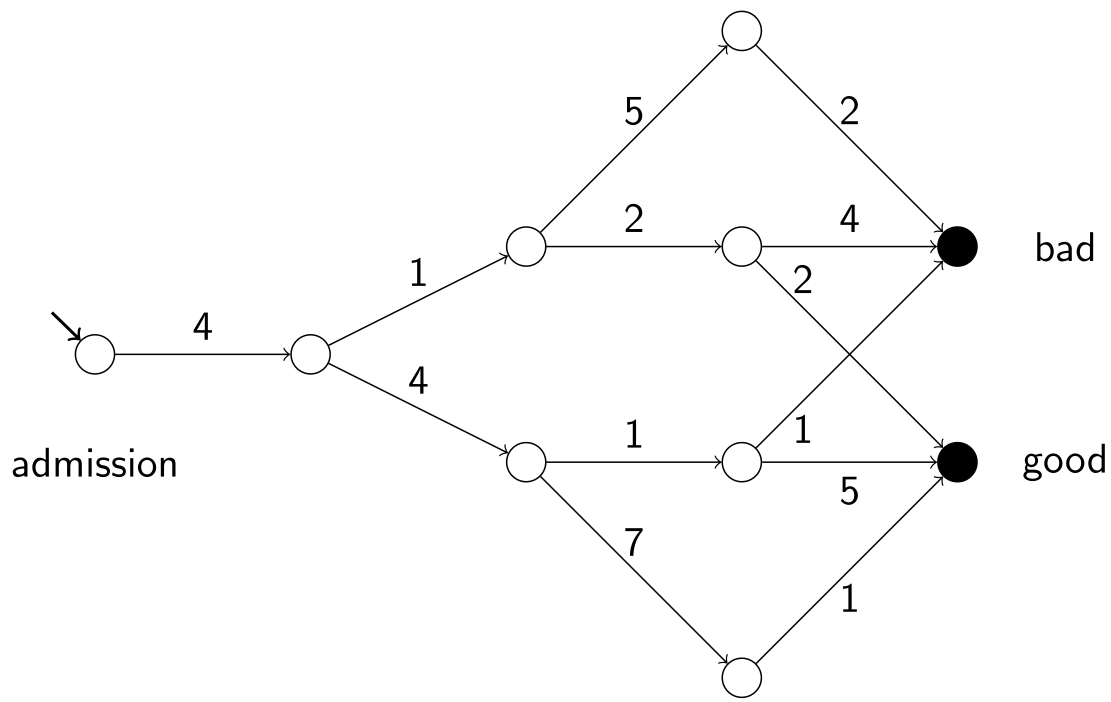
The Q-matrix

$$Q = \begin{pmatrix} A & L \\ 0 & 0 \end{pmatrix}$$

where $A \in \mathbb{R}^{8\times 8}$ and $L \in \mathbb{R}^{8\times 2}$
In particular, for some initial probability $\alpha \in \mathbb{R}^8$

$$f = \begin{pmatrix} f_{bad} & f_{good} \end{pmatrix} = \alpha e^{At} L$$

There are two ways out of the Hospital



$$(A \mid L) = \left(\begin{array}{cccccccc|cc}
-4 & 4 & & & & & & & & \\
 & -5 & 1 & 4 & & & & & & \\
 & & -7 & & 5 & 2 & & & & \\
 & & & & -2 & & & & & 2 \\
 & & & & & -6 & & & 2 & 4 \\
 & & & -8 & & & 1 & 7 & & \\
 & & & & & & -6 & & 5 & 1 \\
 & & & & & & & -1 & 1 &
\end{array}\right)$$

$$= \left(A \mid \vec{L}_{good} \quad \vec{L}_{bad}\right)$$

$$f = \left(f_{bad} \quad f_{good}\right)$$

$$= \left(\alpha e^{At} \vec{L}_{bad} \quad \alpha e^{At} \vec{L}_{good}\right)$$

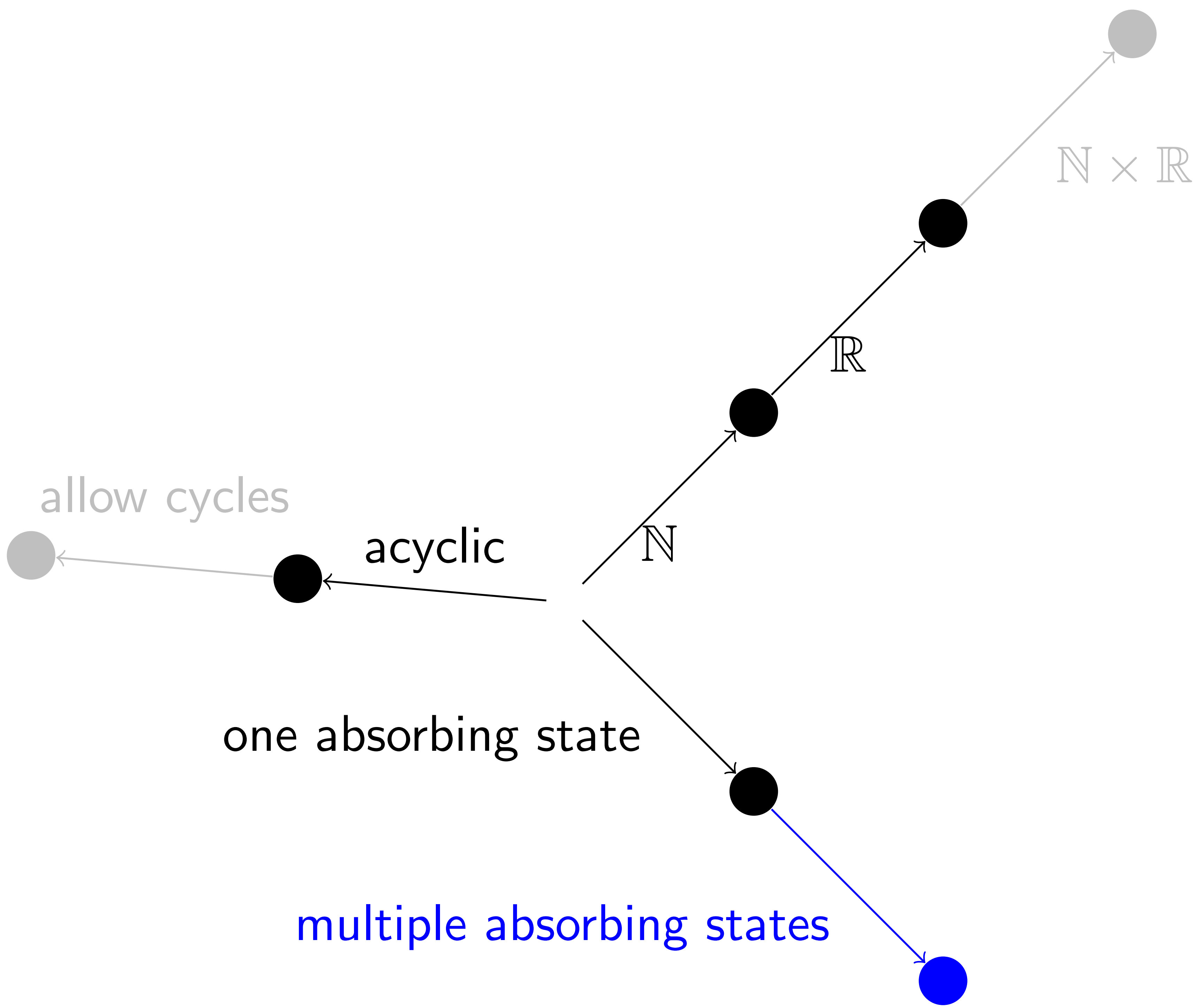
$$= \alpha e^{At} L$$

Stochastics

$$f_{good}(t) = \mathbb{P}\left(X = t \cap \text{Abs} = \text{good}\right)$$

$$\begin{aligned}\mathbb{P}(\text{Abs} = \text{good}) &= \int_{\mathbb{R}} \mathbb{P}\left(X = t \cap \text{Abs} = \text{good}\right)\, dt \\ &= \alpha(-A)^{-1}\vec{L}_{good}\end{aligned}$$

Digging Even Deeper



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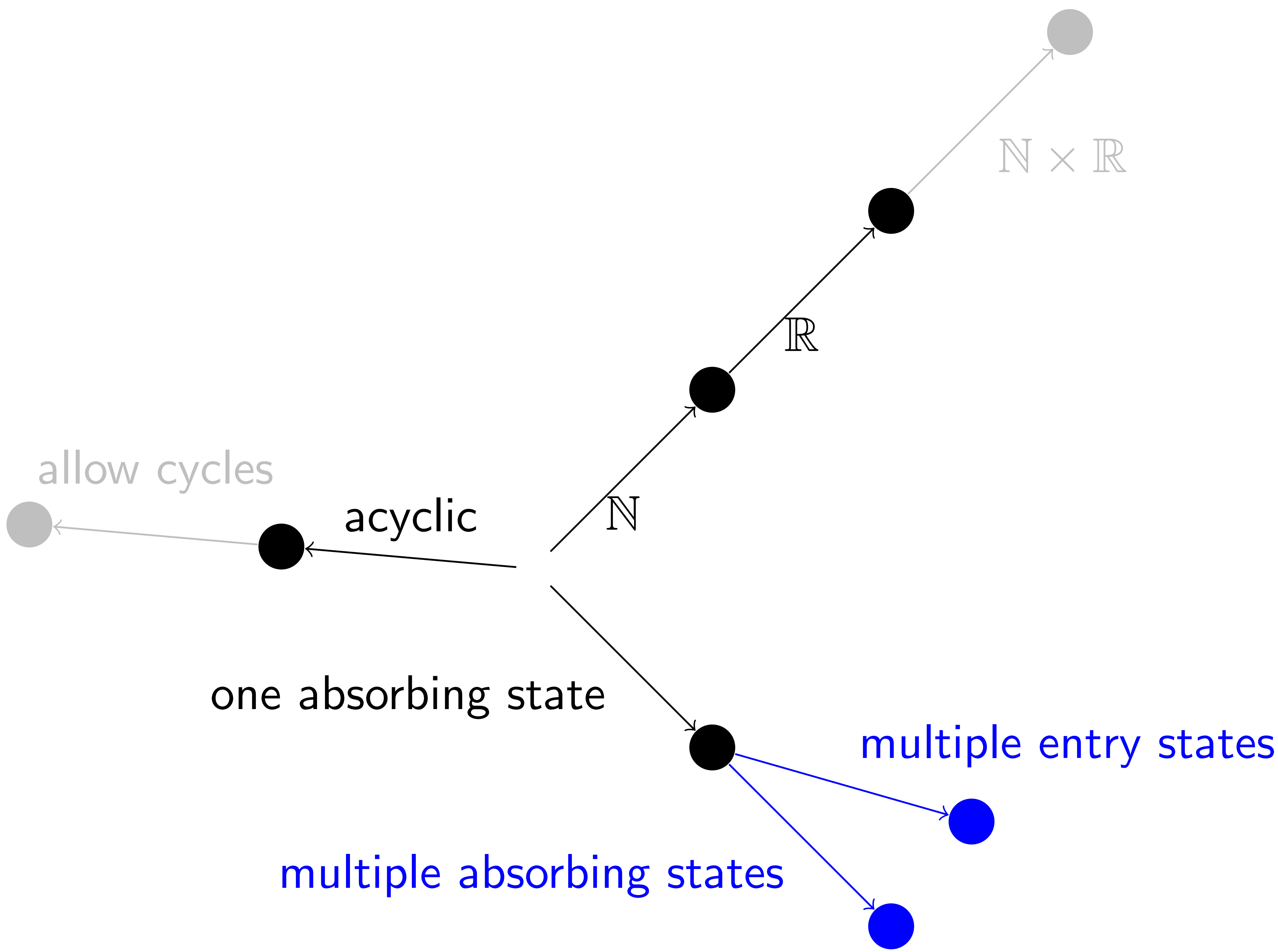
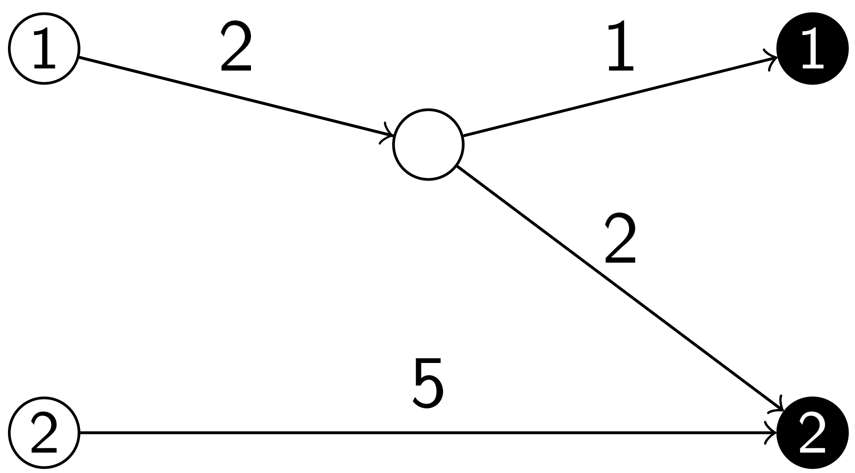


Multiple Entries and Exits



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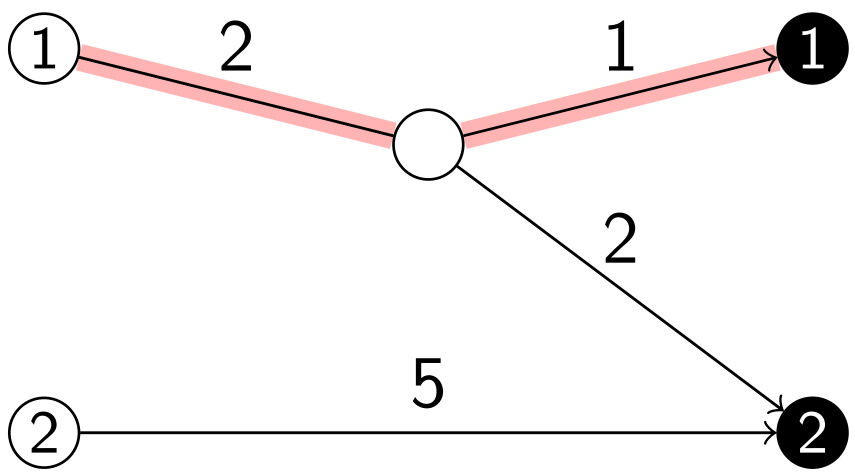

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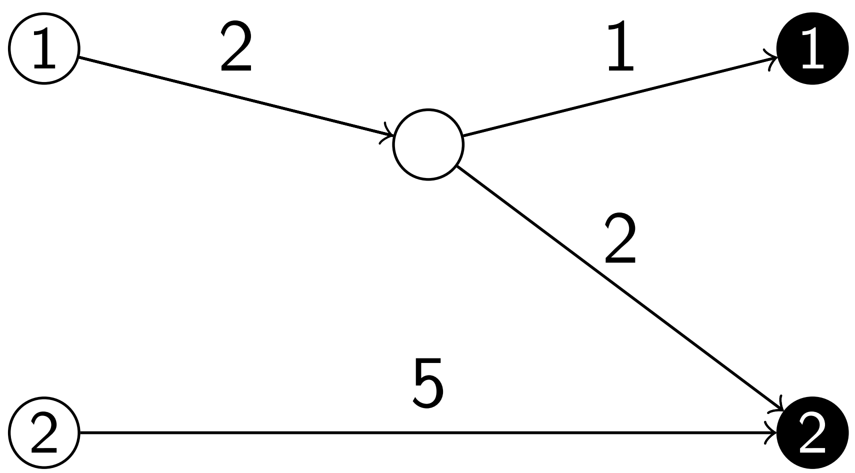
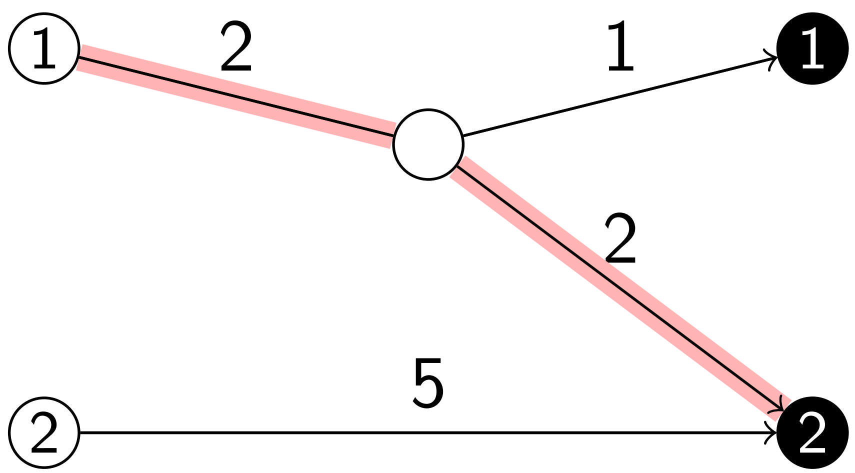


$$f_{1\to 1} = \frac{1}{3}\mathsf{Exp}(2) * \mathsf{Exp}(3)$$

Multiple Entries and Exits



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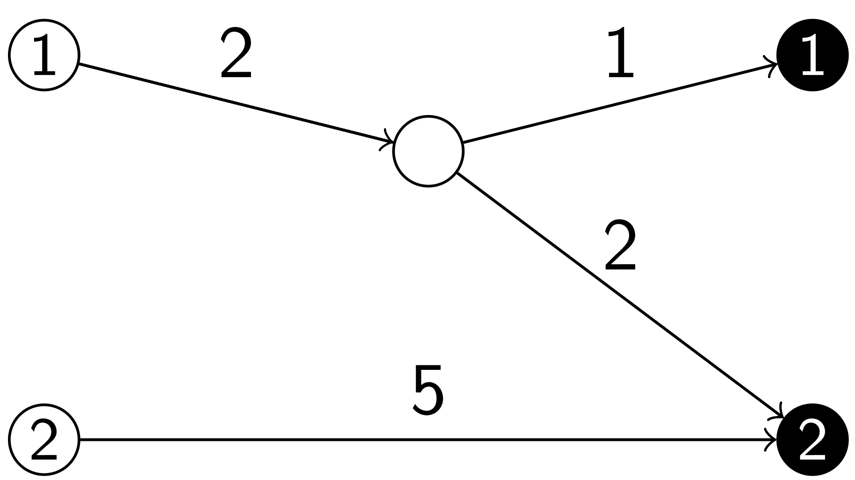
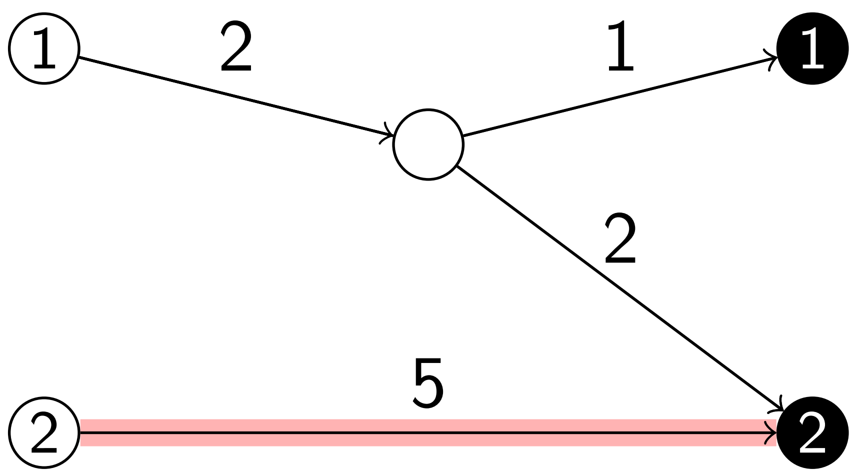
Multiple Entries and Exits



$$f_{1\to 1} = \frac{1}{3}\mathrm{Exp}(2) * \mathrm{Exp}(3)$$

$$f_{1\to 2} = \frac{2}{3}\mathrm{Exp}(2) * \mathrm{Exp}(3)$$

Multiple Entries and Exits



$$f_{1\to 1} = \frac{1}{3}\mathrm{Exp}(2) * \mathrm{Exp}(3)$$

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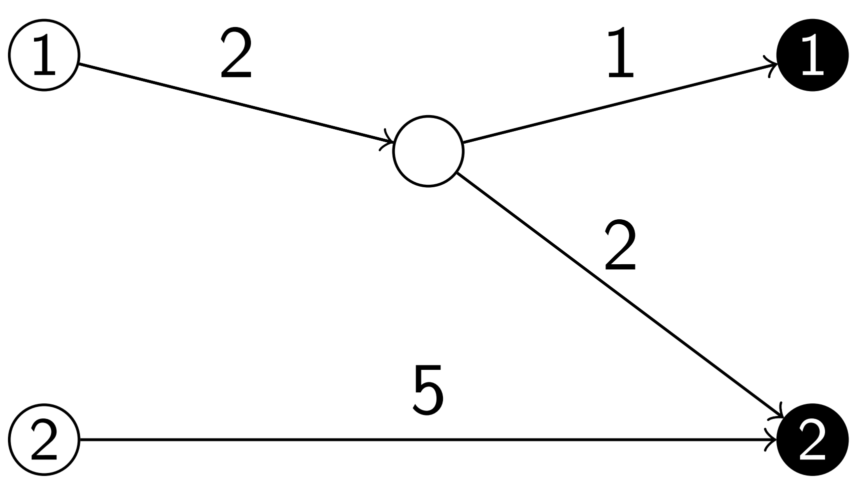
Multiple Entries and Exits

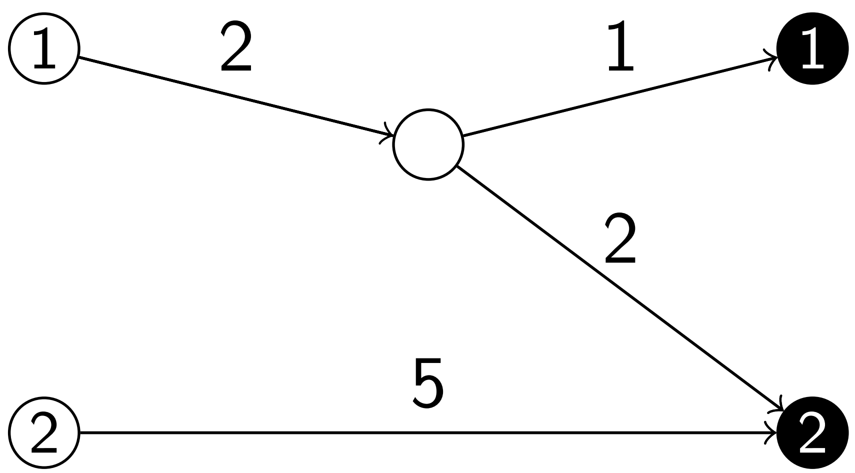


$$f_{1\to 1} = \frac{1}{3}\mathsf{Exp}(2) * \mathsf{Exp}(3)$$

$$f_{1\to 2} = \frac{2}{3}\mathsf{Exp}(2) * \mathsf{Exp}(3)$$

$$f_{2\to 2} = \mathsf{Exp}(5)$$

Multiple Entries and Exits



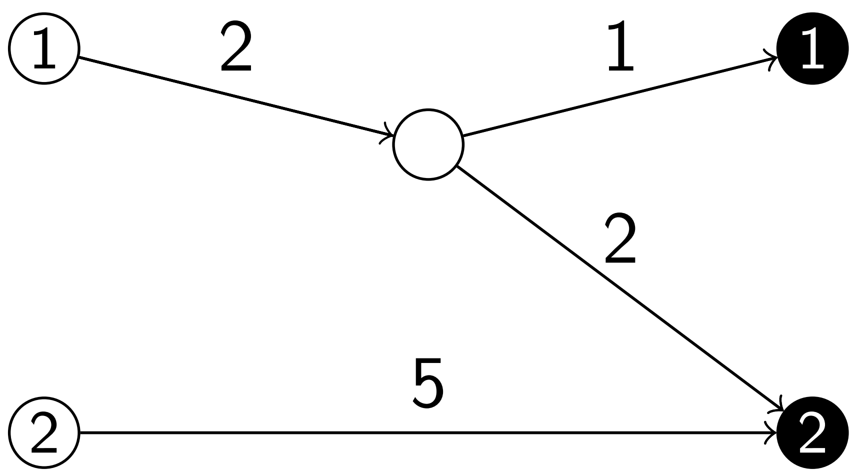
$$f_{1\to 1} = \frac{1}{3}\mathrm{Exp}(2) * \mathrm{Exp}(3)$$

$$f_{1\to 2} = \frac{2}{3}\mathrm{Exp}(2) * \mathrm{Exp}(3)$$

$$f_{2\to 1} = 0$$

$$f_{2\to 2} = \mathrm{Exp}(5)$$

Multiple Entries and Exits



$$f_{1\to 1} = \frac{1}{3}\mathrm{Exp}(2) * \mathrm{Exp}(3)$$

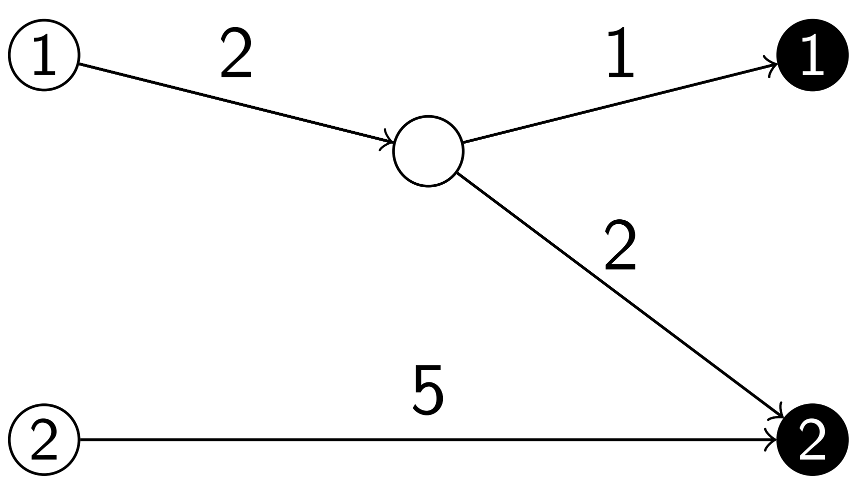
$$f_{1\to 2} = \frac{2}{3}\mathrm{Exp}(2) * \mathrm{Exp}(3)$$

$$f_{2\to 1} = 0$$

$$f_{2\to 2} = \mathrm{Exp}(5)$$

- At ①, behave like the Mexit $\begin{pmatrix} f_{1\to 1} & f_{1\to 2} \end{pmatrix}$
- At ②, behave like the Mexit $\begin{pmatrix} f_{2\to 1} & f_{2\to 2} \end{pmatrix}$

Multiple Entries and Exits



$$f_{1\to 1} = \frac{1}{3}\text{Exp}(2) * \text{Exp}(3)$$

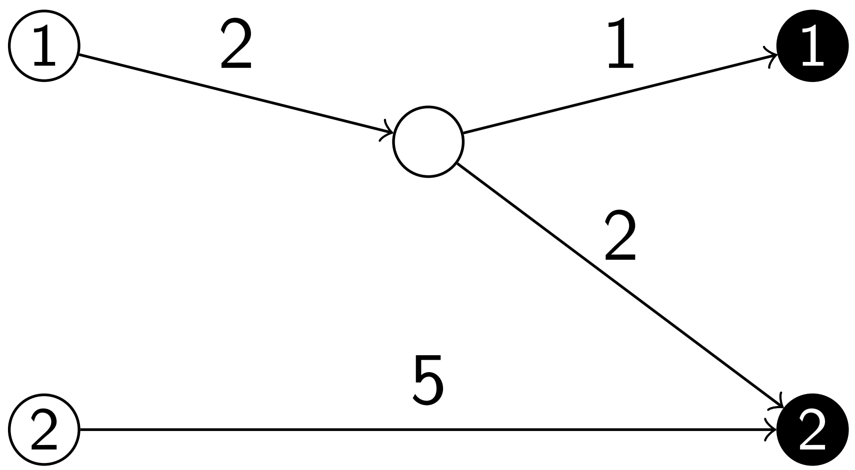
$$f_{1\to 2} = \frac{2}{3}\text{Exp}(2) * \text{Exp}(3)$$

$$f_{2\to 1} = 0$$

$$f_{2\to 2} = \text{Exp}(5)$$

- At ①, behave like the Mexit $\begin{pmatrix} f_{1\to 1} & f_{1\to 2} \end{pmatrix} = f_{1\to\bullet}$
- At ②, behave like the Mexit $\begin{pmatrix} f_{2\to 1} & f_{2\to 2} \end{pmatrix} = f_{2\to\bullet}$

Multiple Entries and Exits



$$F = \begin{pmatrix} f_{1\to\bullet} \\ f_{2\to\bullet} \end{pmatrix} = \begin{pmatrix} f_{1\to 1} & f_{1\to 2} \\ f_{2\to 1} & f_{2\to 2} \end{pmatrix}$$

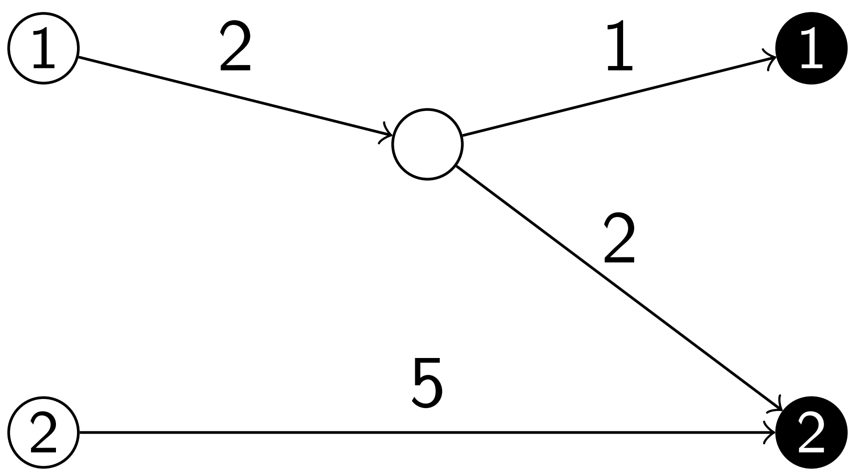
$$(F)_{ij} = \mathbb{P}\left(X = t, A = j \mid S = i\right)$$

Given a probability distribution α over $\{①, ②\}$

$$\alpha = \big(\mathbb{P}(Start = ①) \quad \mathbb{P}(Start = ②)\big)$$

we can compute the Mexit F at α by αF.

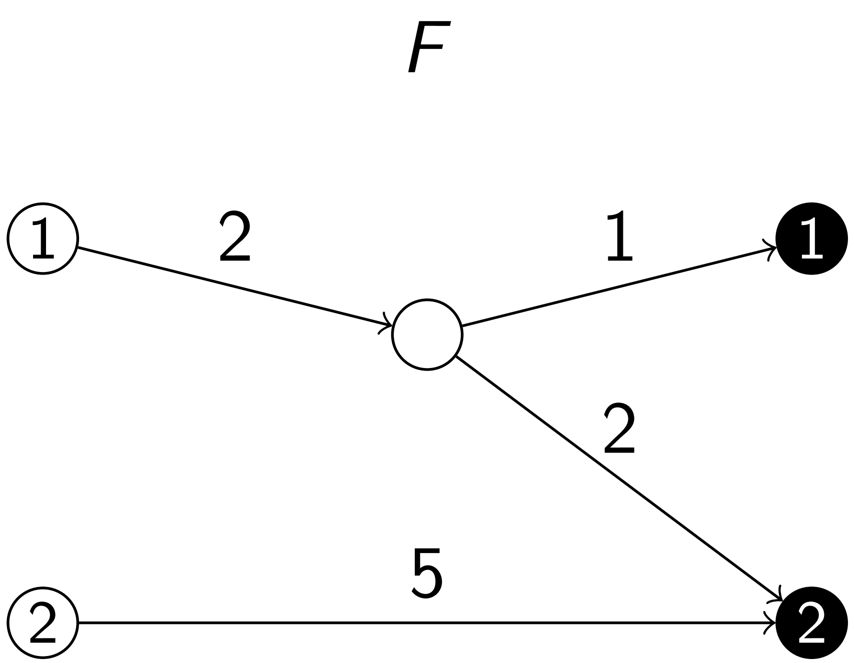
Example

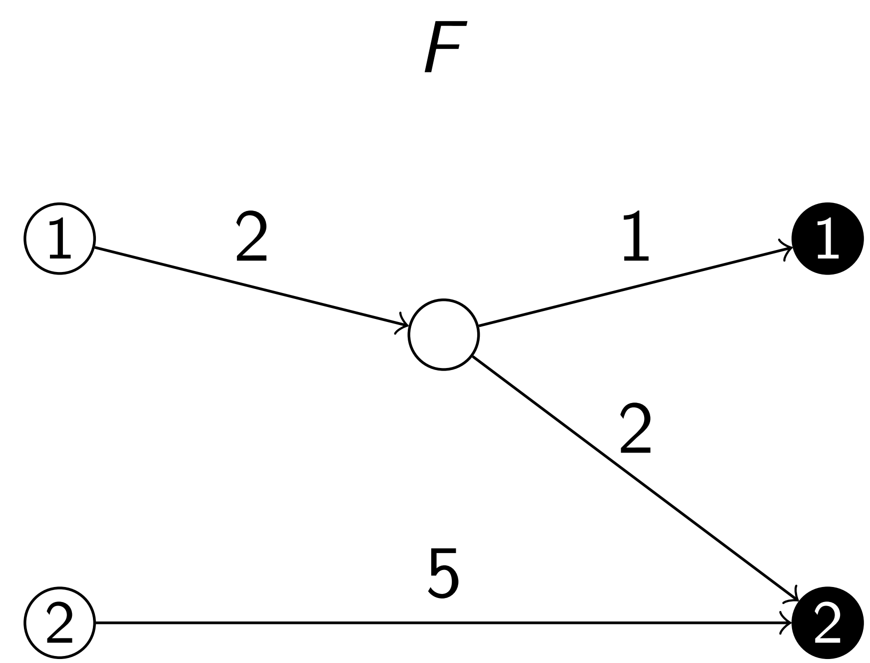


With $\alpha = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \end{pmatrix}$, we get

$$\alpha \cdot F = \frac{1}{3} f_{1\to\bullet} + \frac{2}{3} f_{2\to\bullet} = \begin{pmatrix} \frac{1}{3} f_{1\to 1} + \frac{2}{3} f_{2\to 1} & \frac{1}{3} f_{1\to 2} + \frac{2}{3} f_{2\to 2} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{9}\mathrm{Exp}(2) * \mathrm{Exp}(3) & \frac{2}{9}\mathrm{Exp}(2) * \mathrm{Exp}(3) + \frac{2}{3}\mathrm{Exp}(5) \end{pmatrix}$$

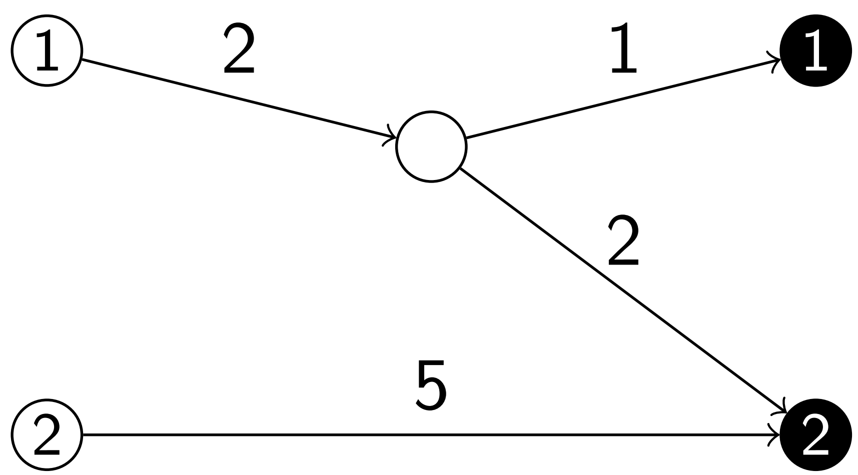
Multiple Multi-Exit Multi-Entry



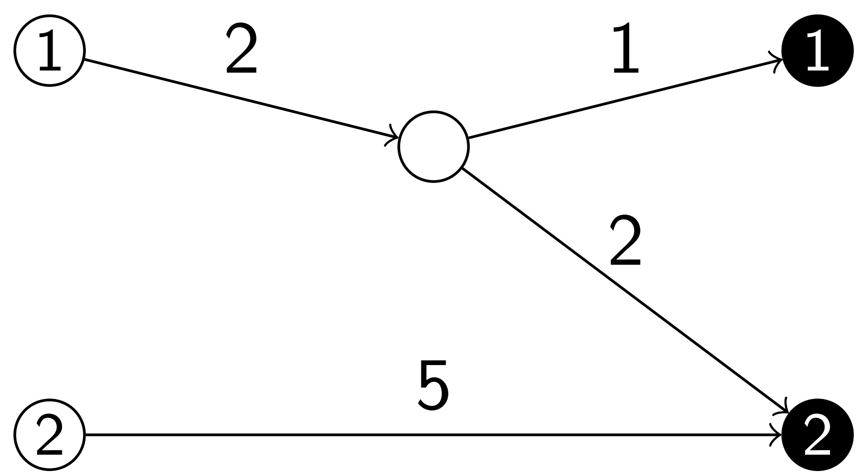


Multiple Multi-Exit Multi-Entry

F



F



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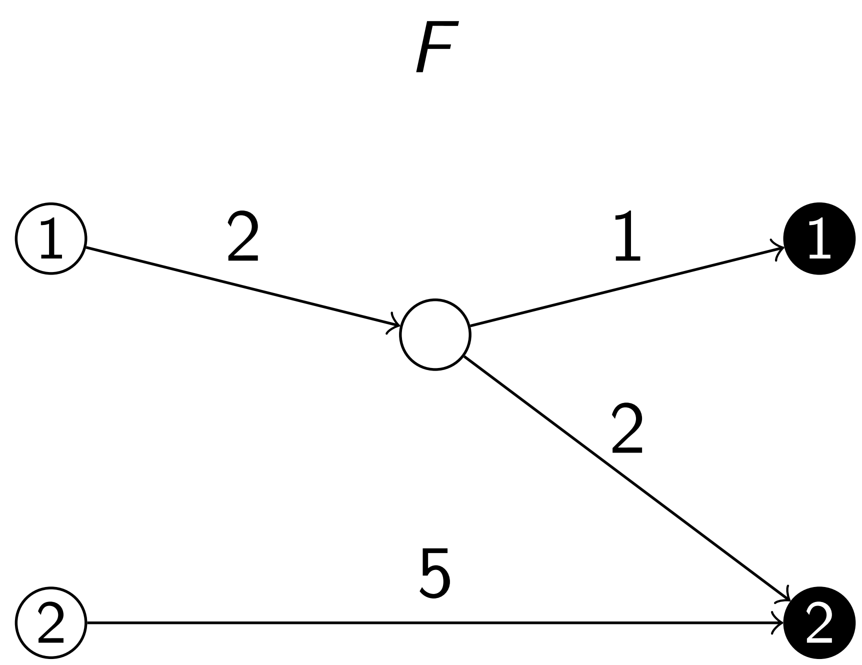
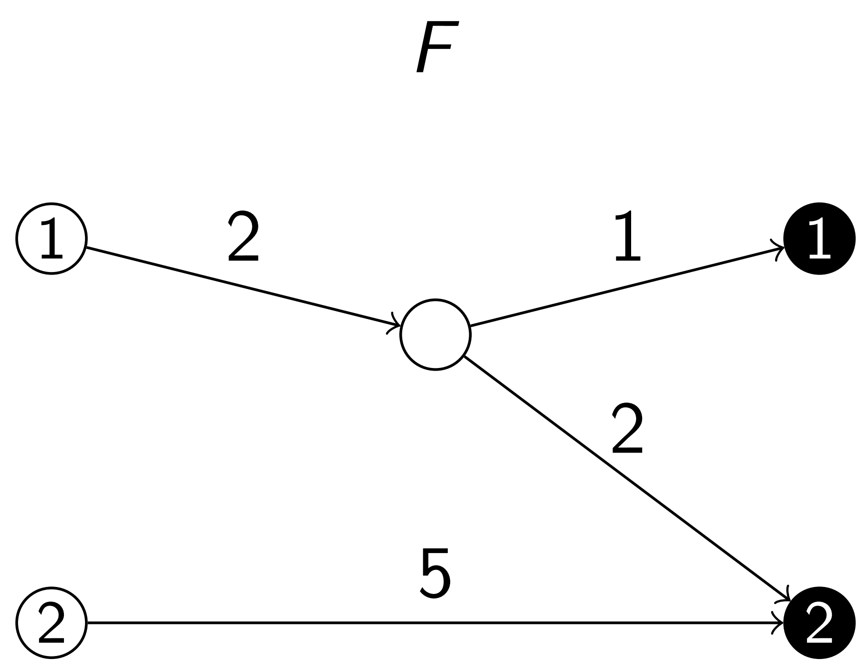
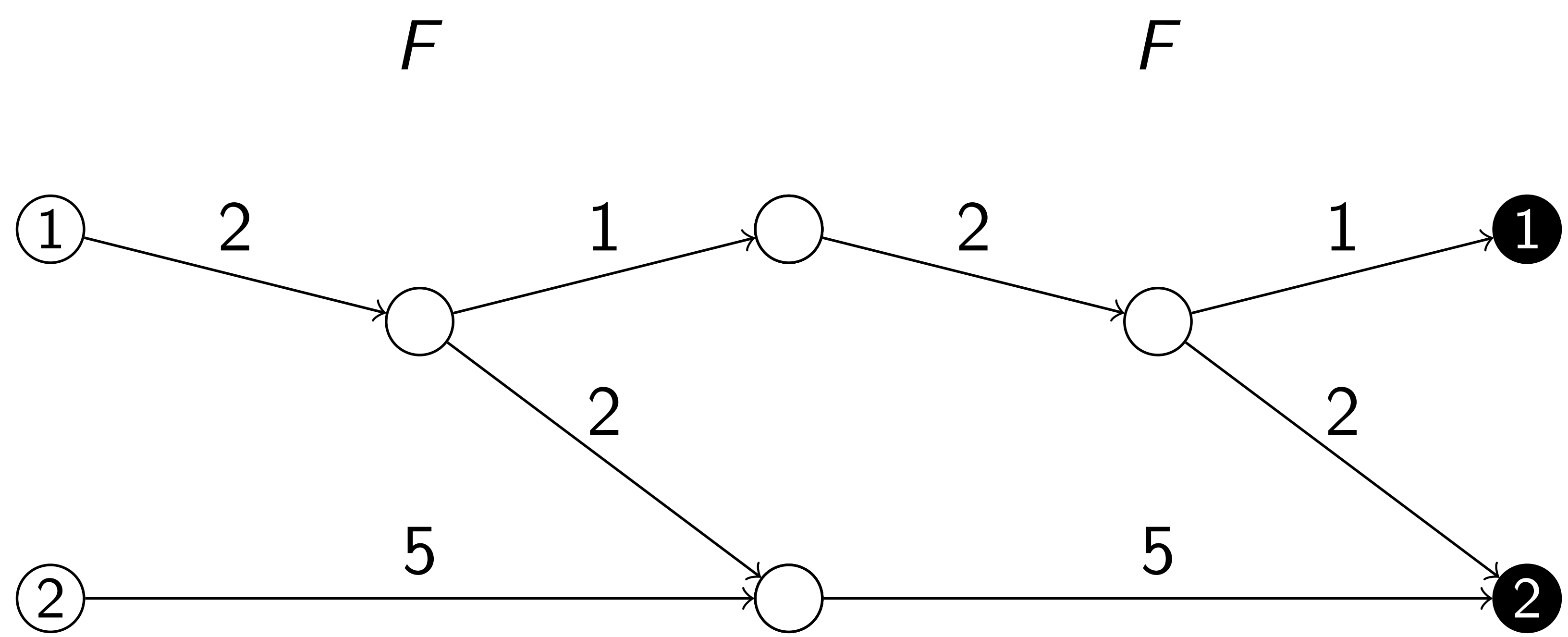
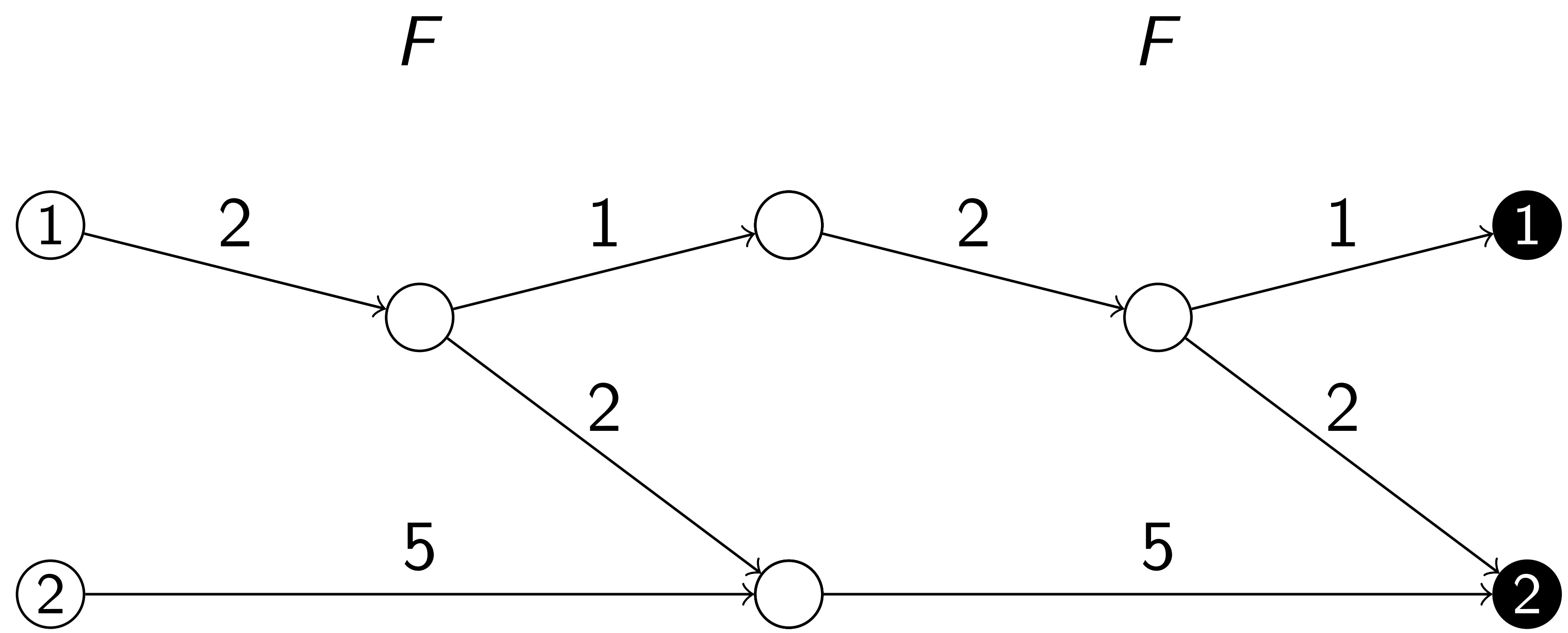




Multiple Multi-Exit Multi-Entry

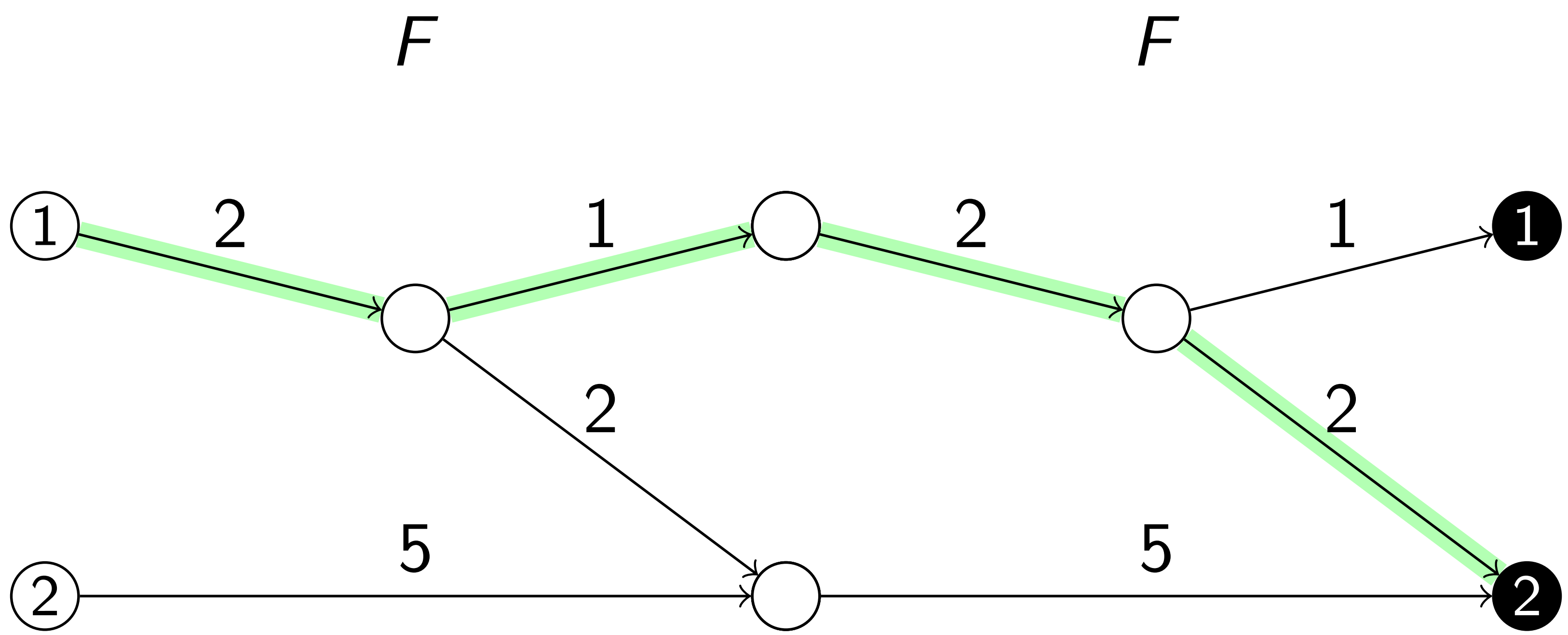


Composition



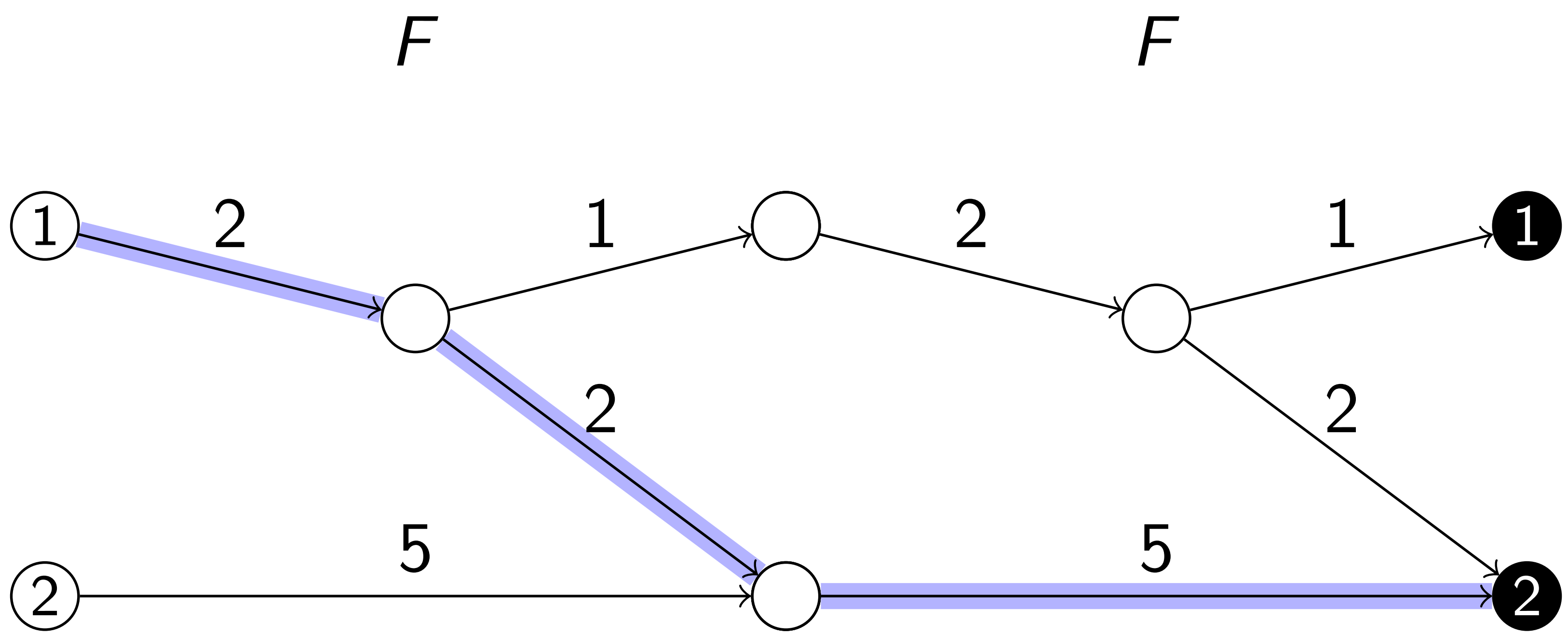
$$(F \circ F)_{1 \to 2} =$$

Composition



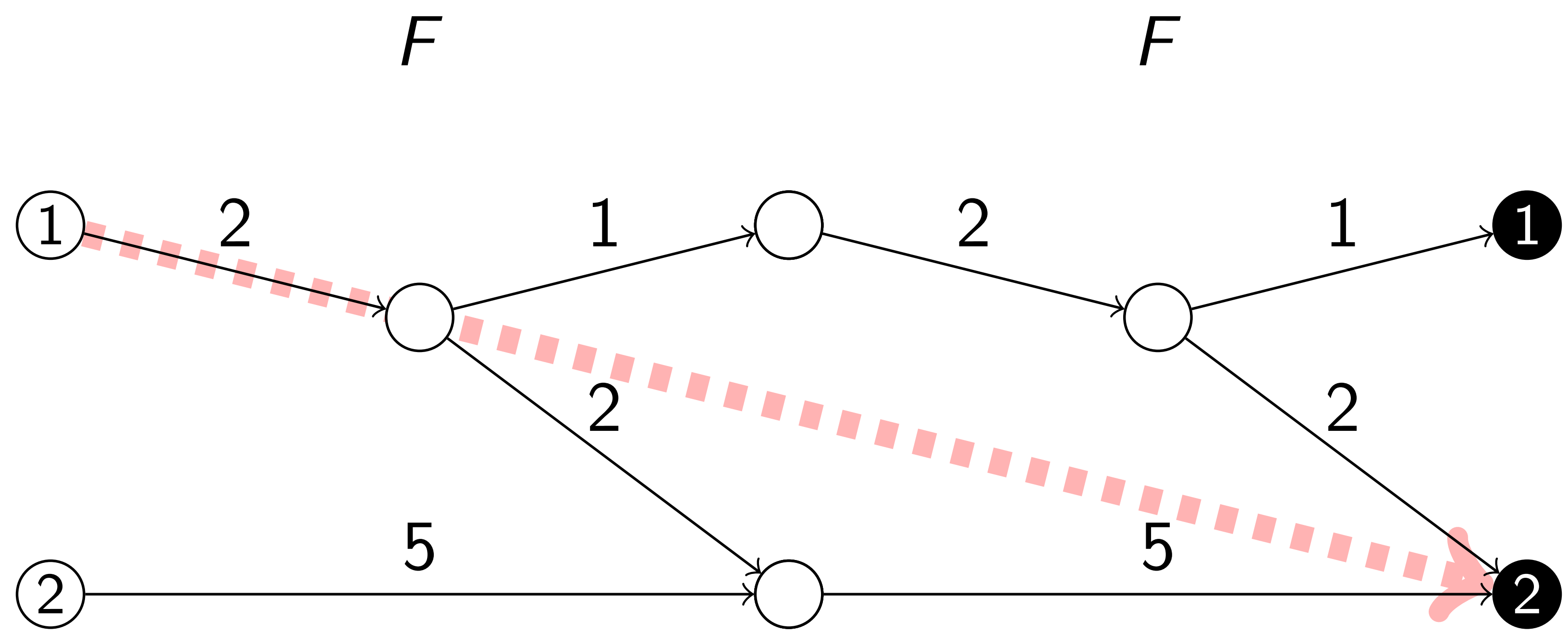
$$(F \circ F)_{1\to 2} = f_{1\to 1} \circ f_{1\to 2} +$$

Composition



$$(F \circ F)_{1 \to 2} = f_{1 \to 1} \circ f_{1 \to 2} + f_{2 \to 1} \circ f_{2 \to 2}$$

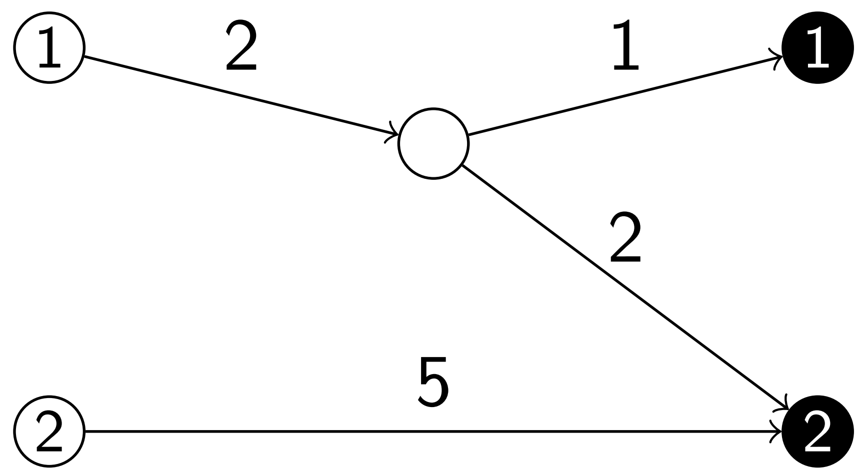
Composition

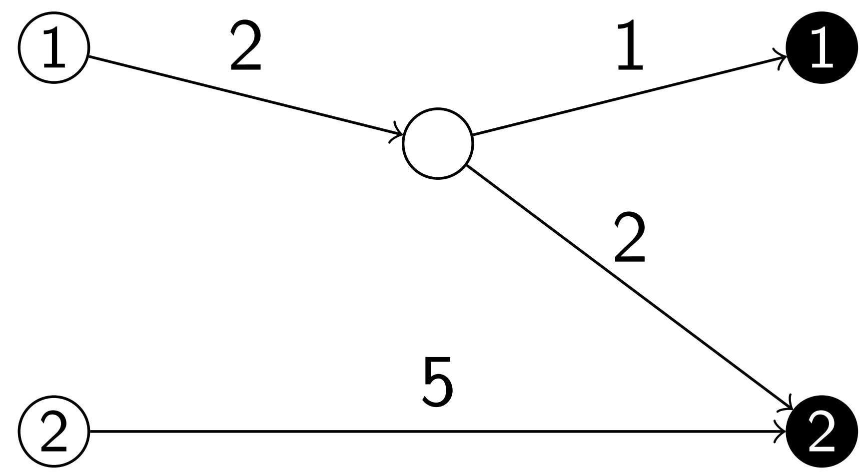


$$(F \circ F)_{1\to 2} = f_{1\to 1} \circ f_{1\to 2} + f_{2\to 1} \circ f_{2\to 2}$$

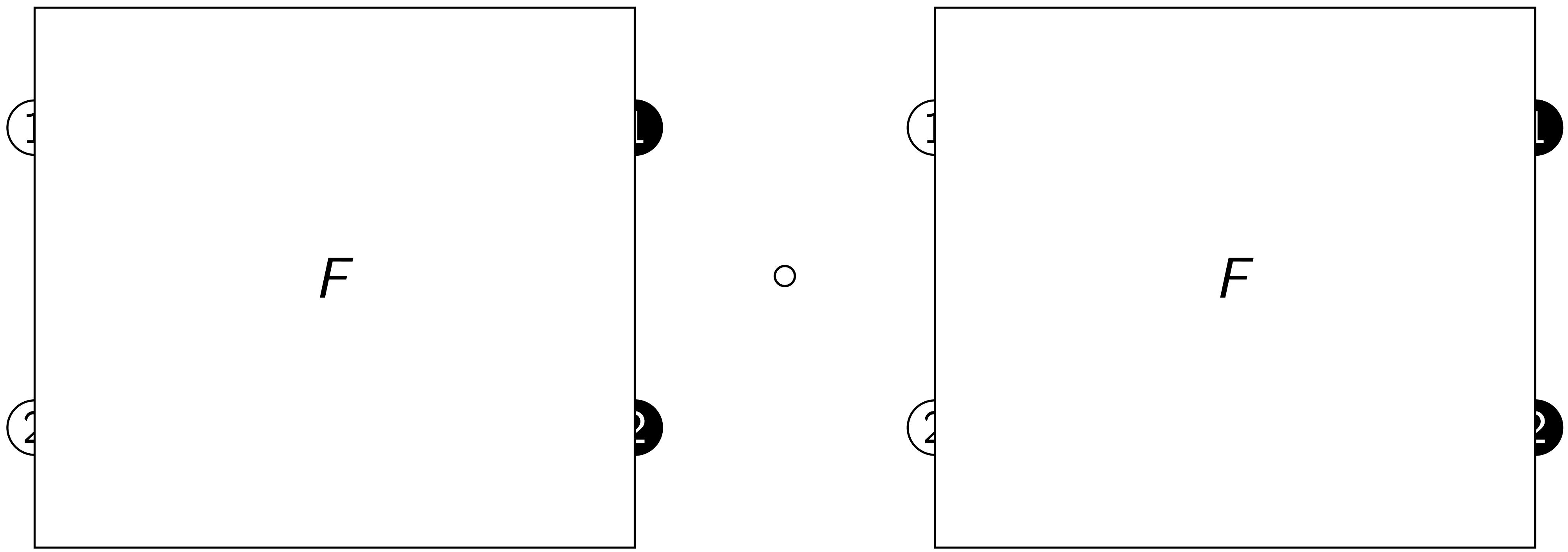
$$= \sum_{k=1}^{2} f_{1\to k} \circ f_{k\to 2}$$

Composition Abstractly

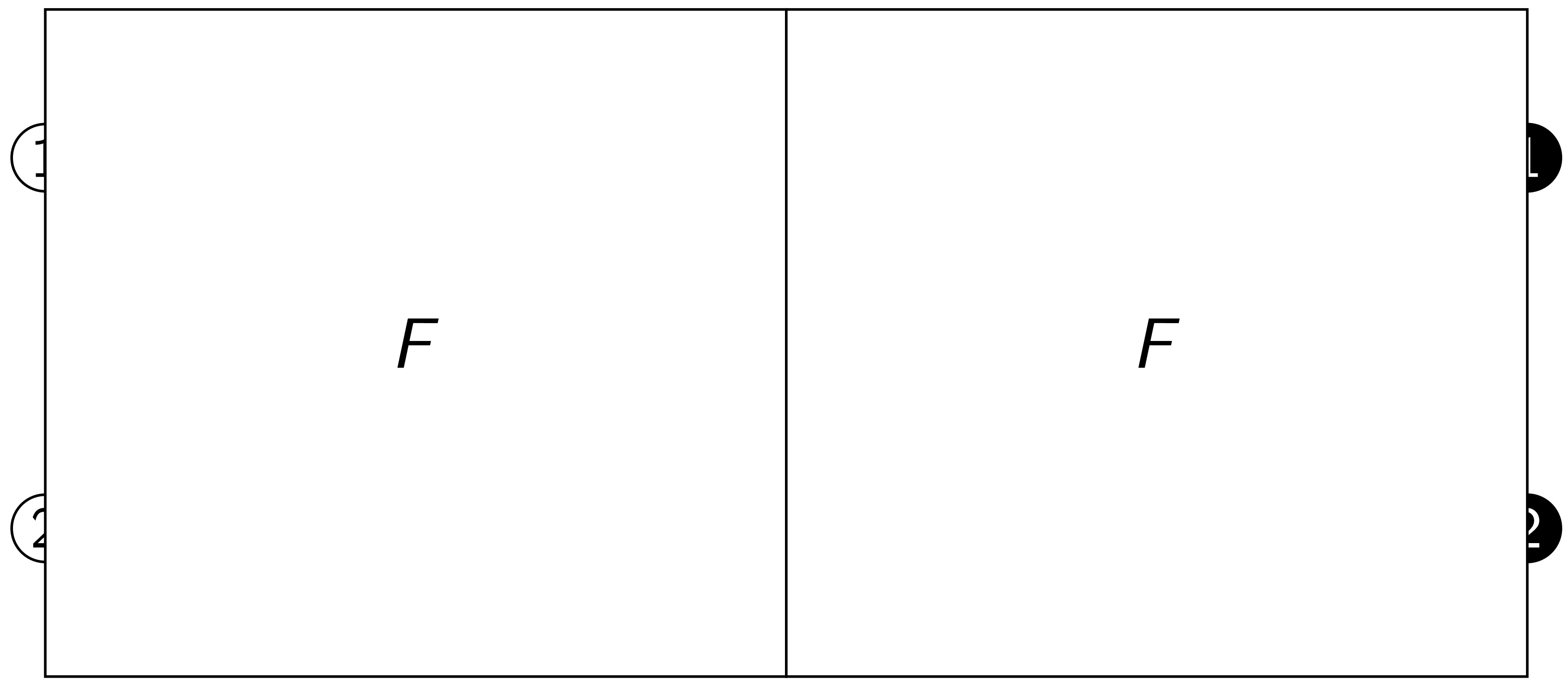




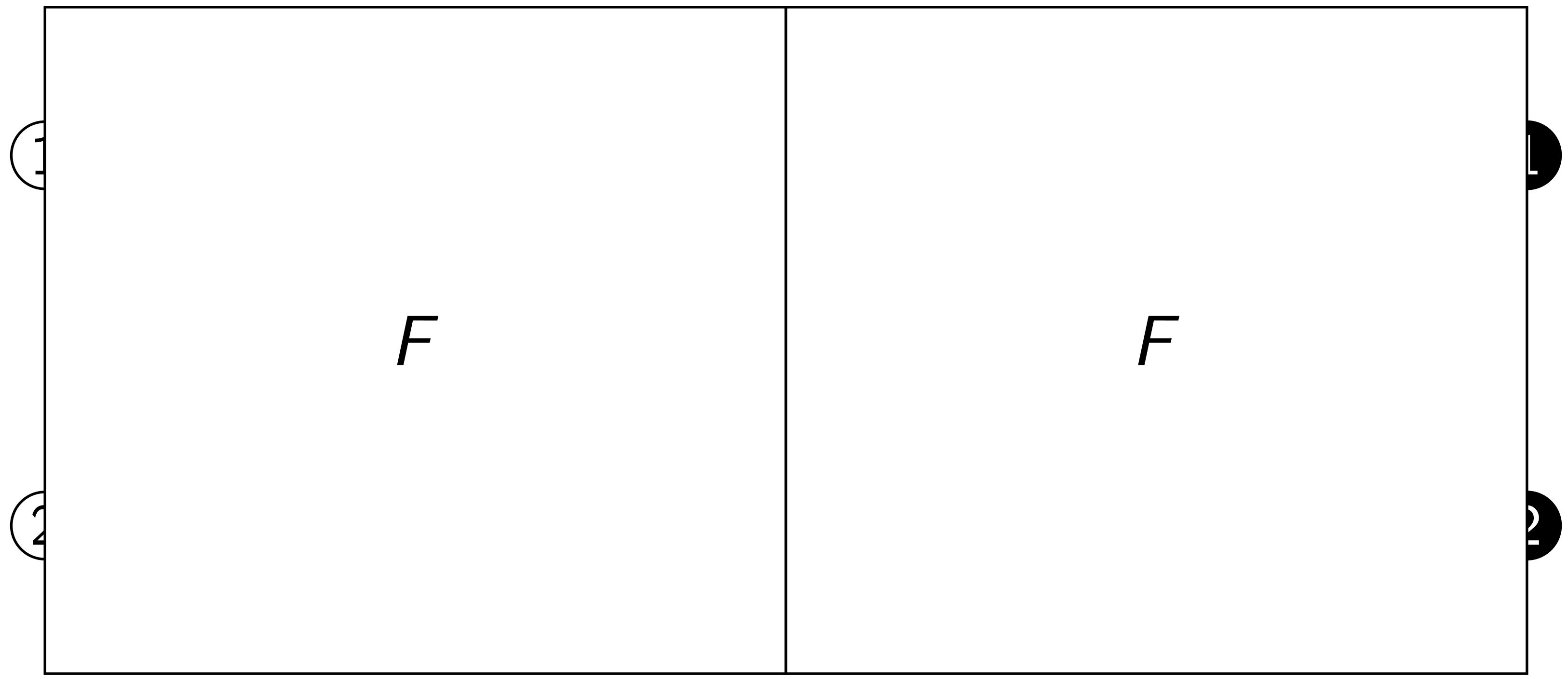
Composition Abstractly



Composition Abstractly

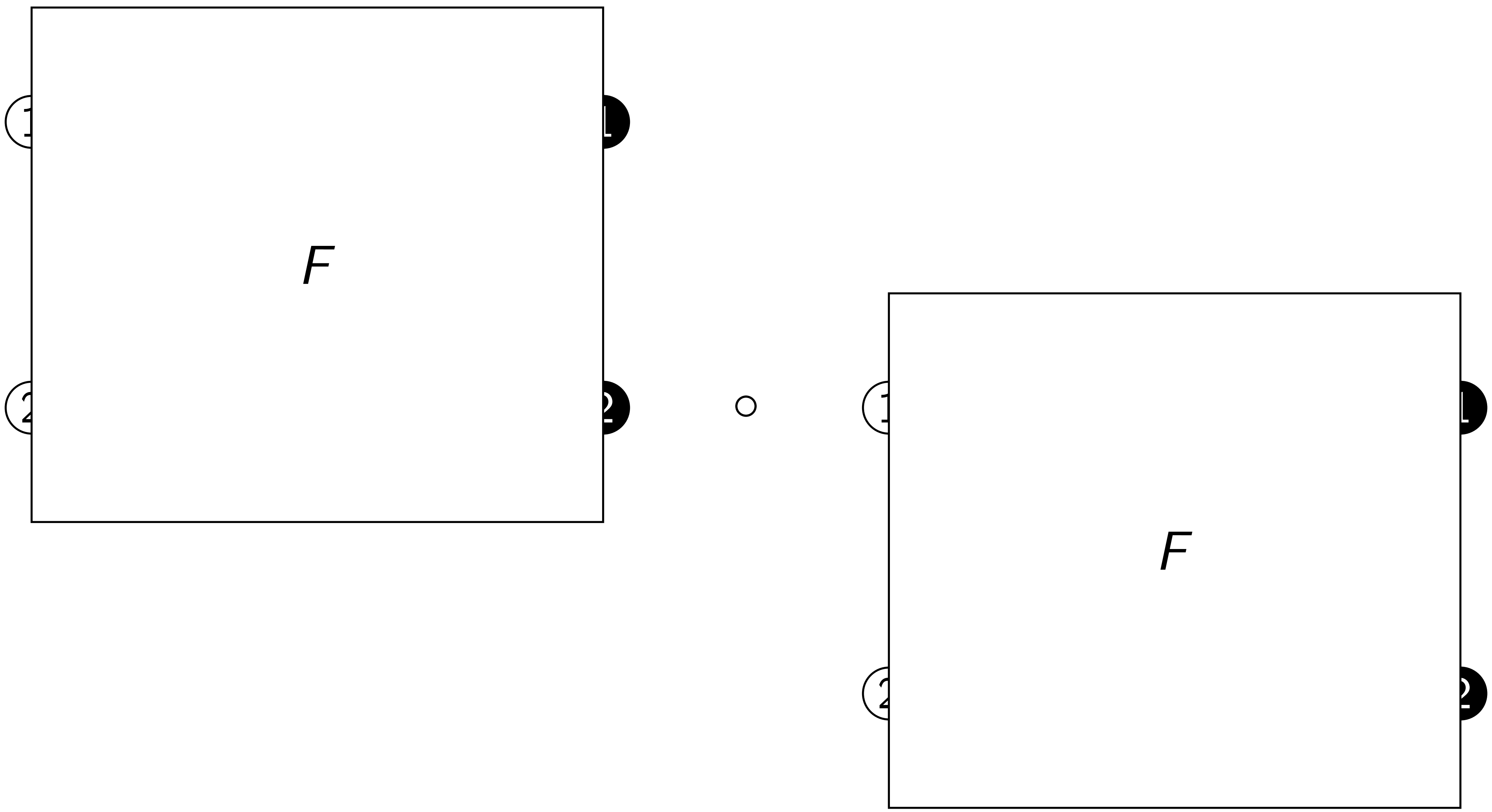


Composition Abstractly

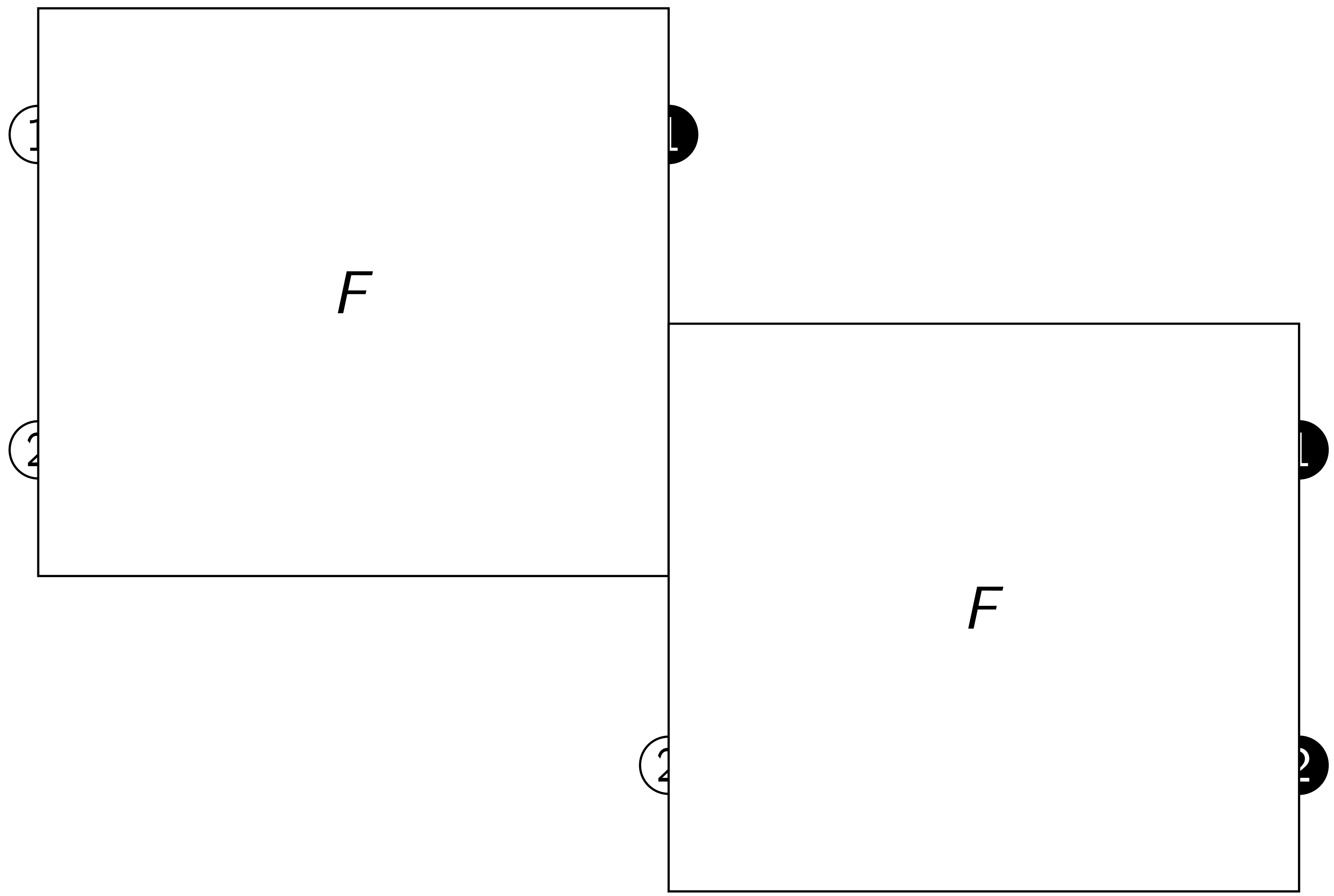


$F \circ F$

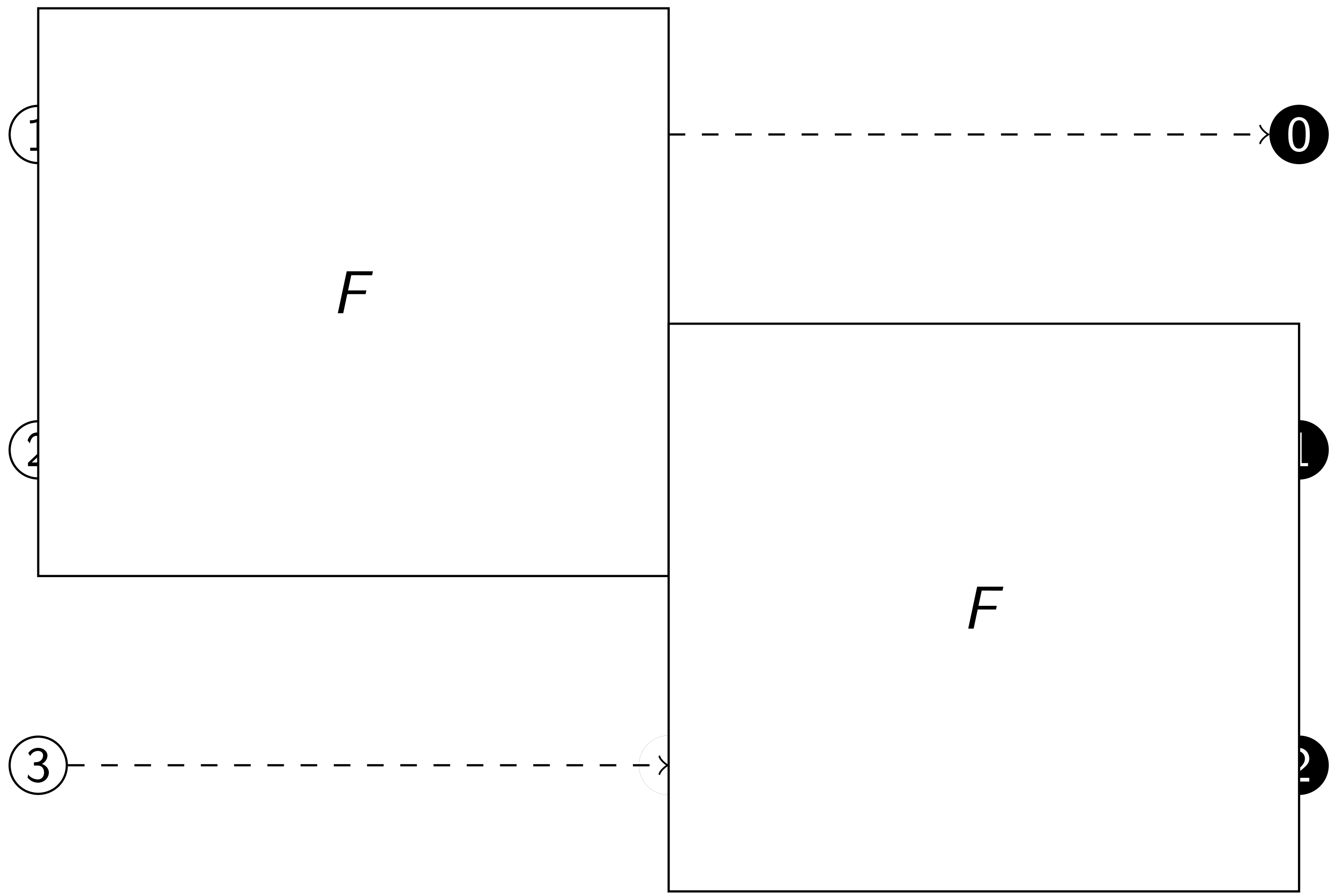
Composition Abstractly



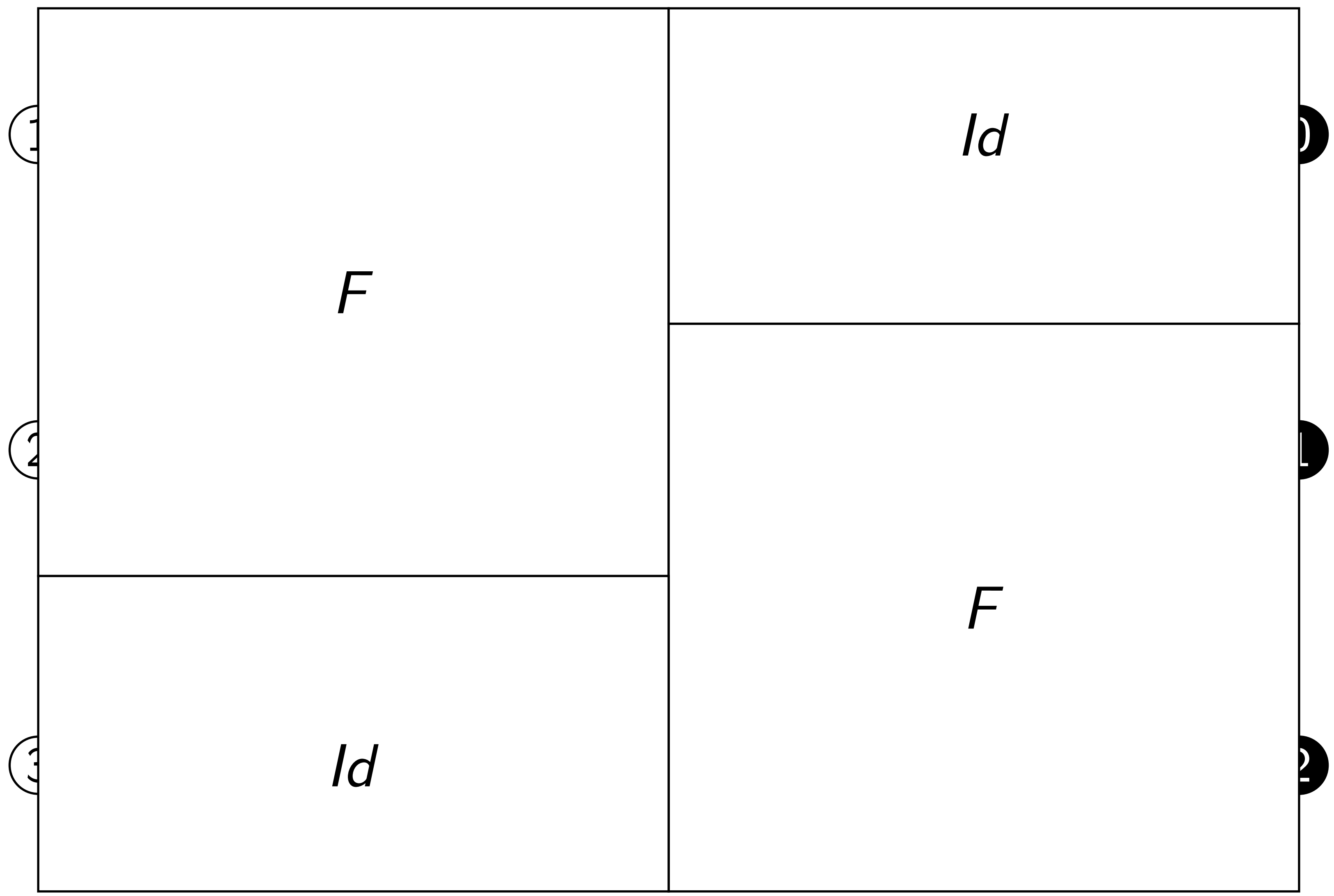
Composition Abstractly



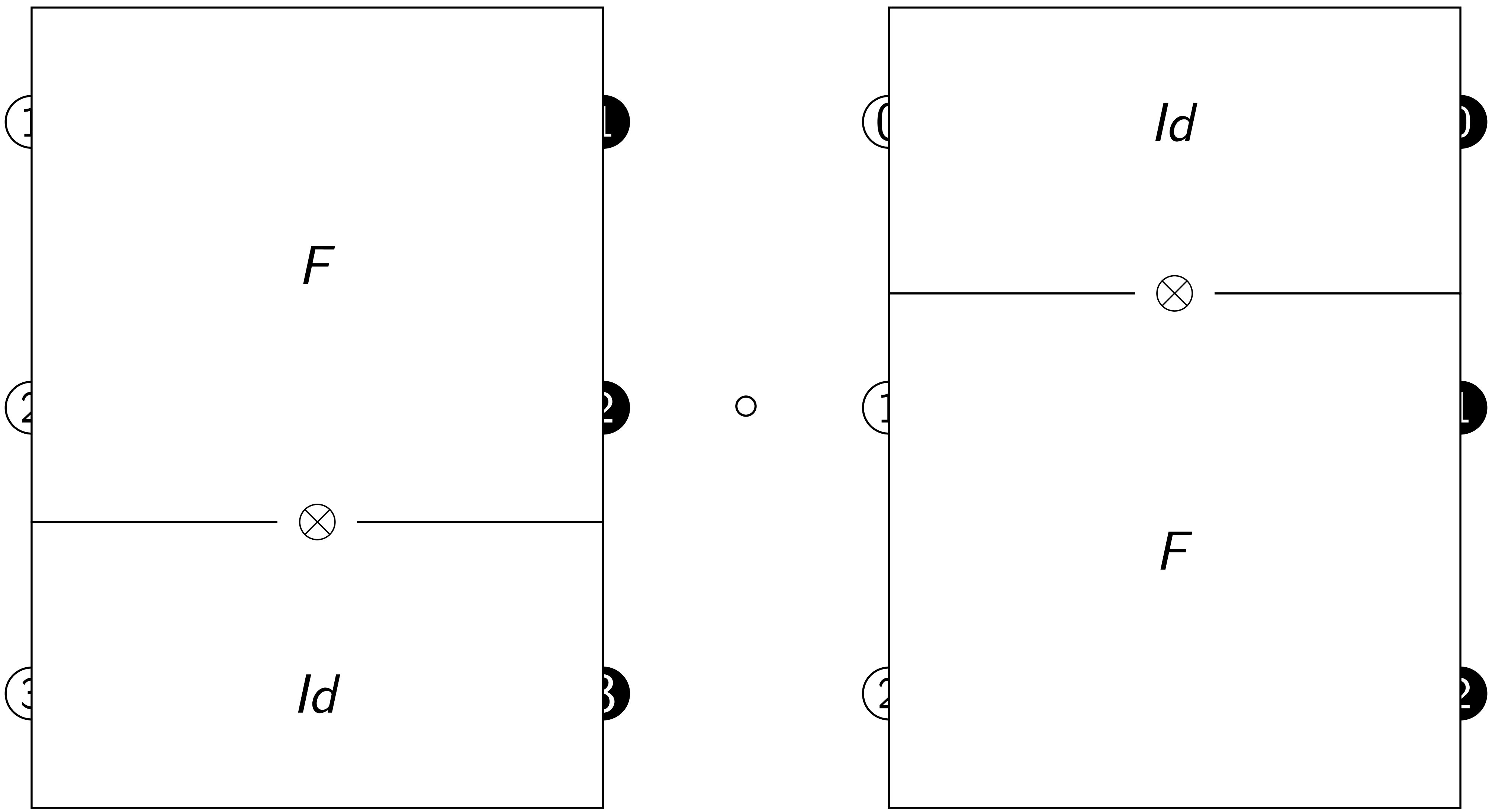
Composition Abstractly



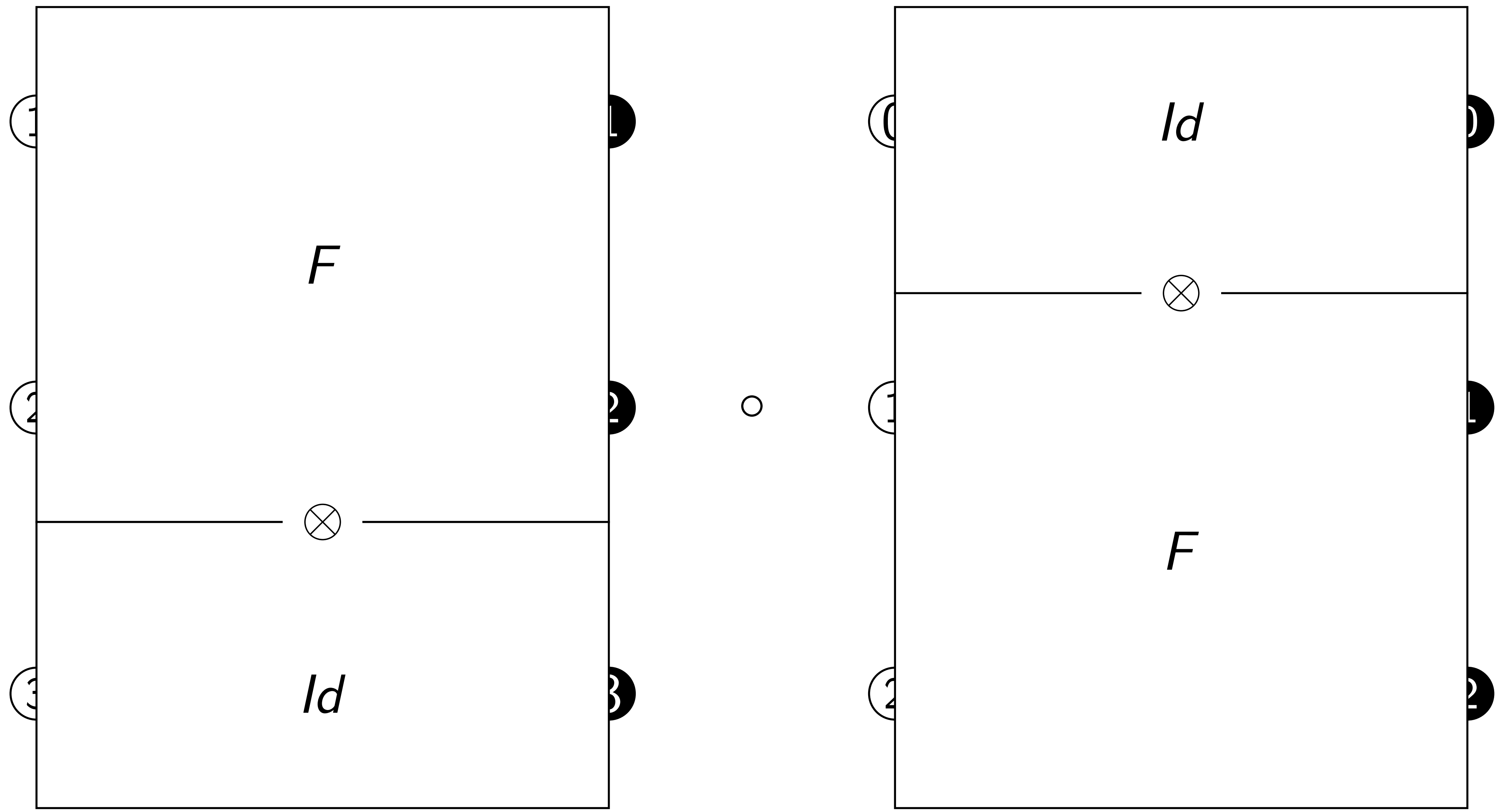
Composition Calculus [4, 3]



Composition Calculus [4, 3]

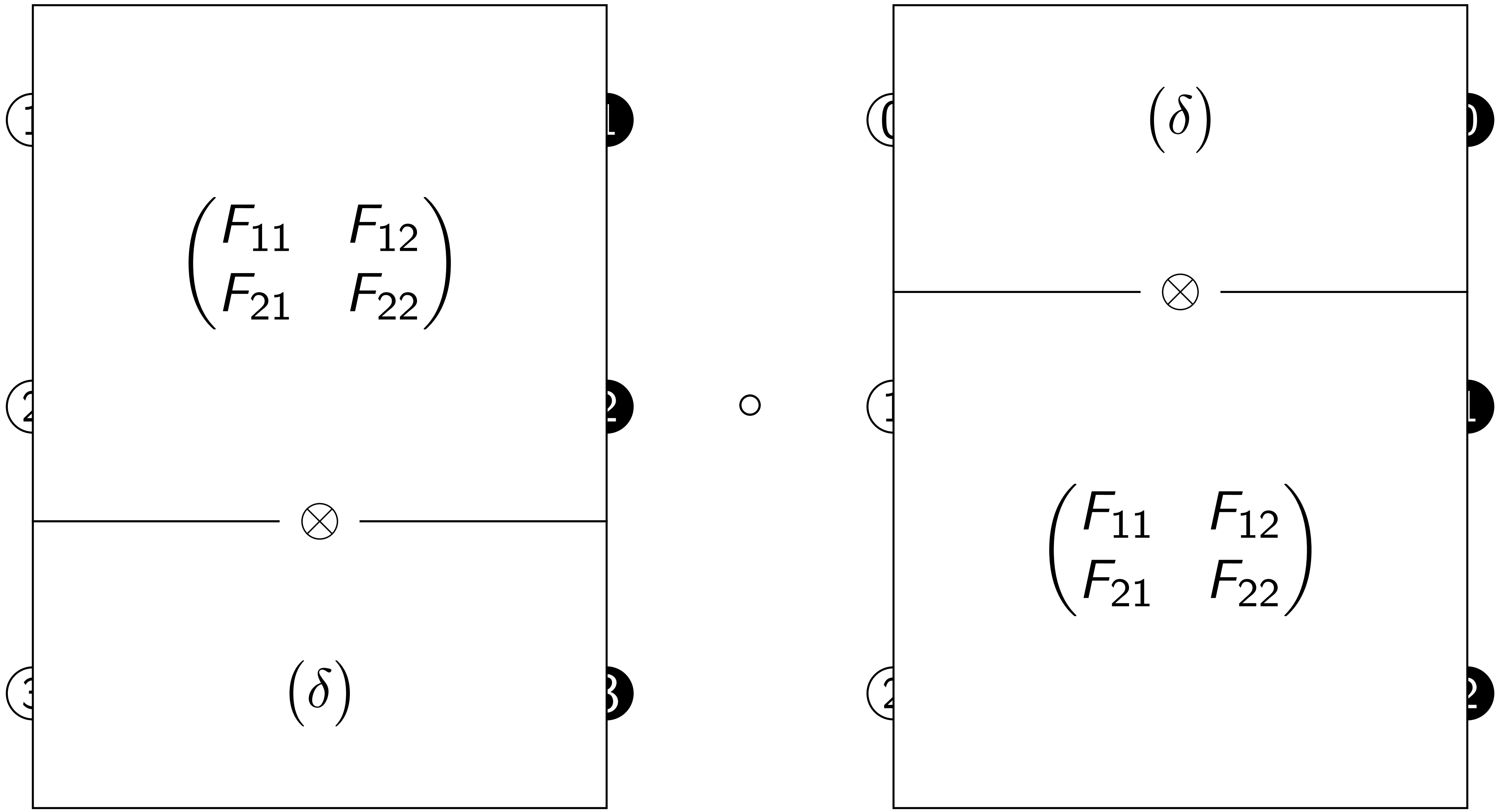


Composition Calculus [4, 3]



$$(F \otimes Id) \circ (Id \otimes F)$$

Composition Calculus [4, 3]



$$(F \otimes Id) \circ (Id \otimes F)$$

Composition Calculus

$$(F \otimes Id) \circ (Id \otimes F)$$

Composition Calculus

$$(A \otimes B) := \left(\begin{array}{c|c} A & \\ \hline & B \end{array} \right)$$

$$(F \otimes Id) \circ (Id \otimes F)$$

Composition Calculus

$$(A \otimes B) := \left(\begin{array}{c|c} A & \\ \hline & B \end{array}\right)$$

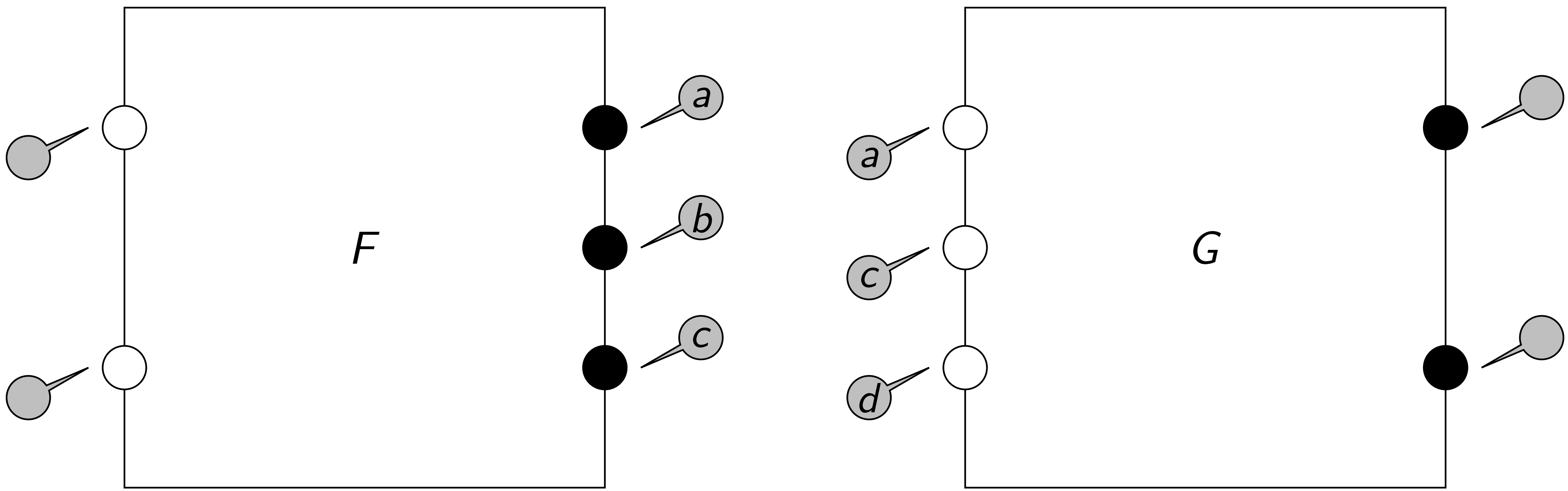
$$(F \otimes Id) \circ (Id \otimes F) = \begin{pmatrix} F_{1\to1} & F_{1\to2} & \\ F_{2\to1} & F_{2\to2} & \\ & & \delta \end{pmatrix} \circ \begin{pmatrix} \delta & & \\ & F_{1\to1} & F_{1\to2} \\ & F_{2\to1} & F_{2\to2} \end{pmatrix}$$

Composition Calculus

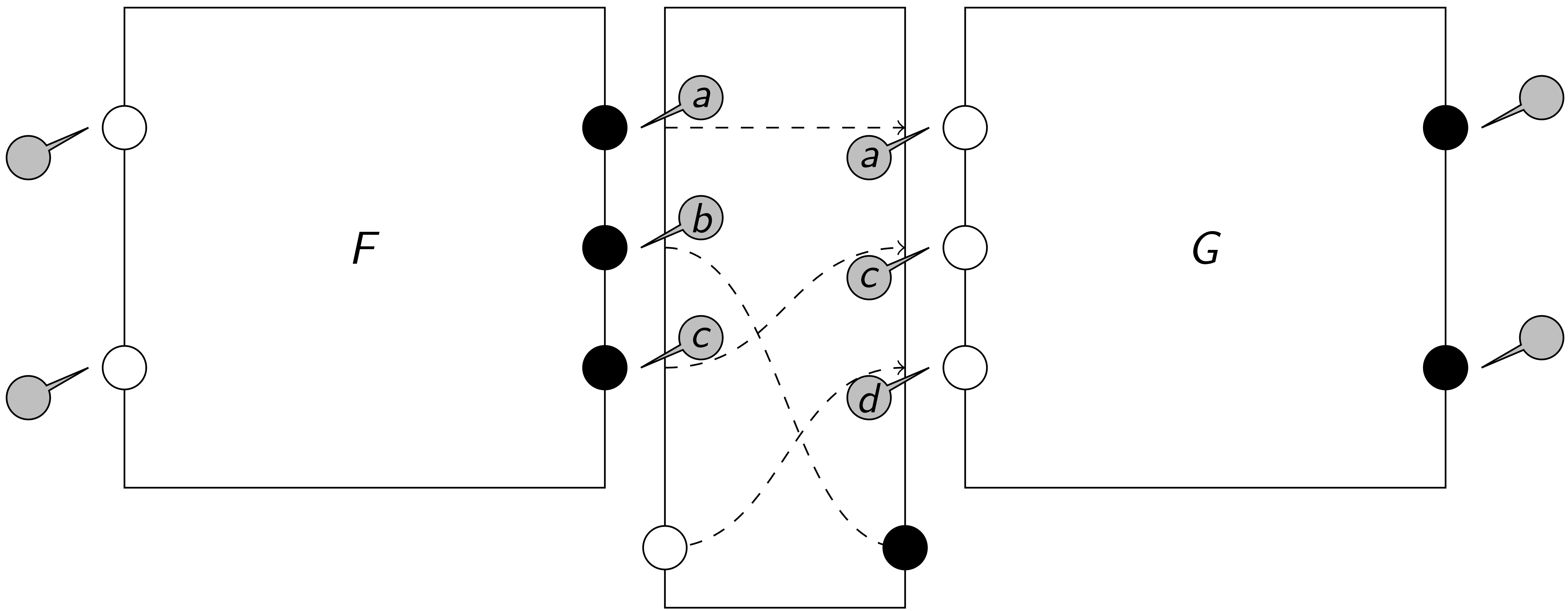
$$(A \otimes B) := \left(\begin{array}{c|c} A & \\ \hline & B \end{array}\right)$$

$$\begin{aligned}(F \otimes Id) \circ (Id \otimes F) &= \begin{pmatrix} F_{1\to 1} & F_{1\to 2} & \\ F_{2\to 1} & F_{2\to 2} & \\ & & \delta \end{pmatrix} \circ \begin{pmatrix} \delta & & \\ & F_{1\to 1} & F_{1\to 2} \\ & F_{2\to 1} & F_{2\to 2} \end{pmatrix} \\ &= \begin{pmatrix} F_{1\to 1} & F_{1\to 1} \circ F_{1\to 2} & F_{1\to 2} \circ F_{1\to 2} \\ F_{2\to 1} & F_{1\to 1} \circ F_{2\to 2} & F_{1\to 2} \circ F_{2\to 2} \\ & F_{2\to 1} & F_{2\to 2} \end{pmatrix}\end{aligned}$$

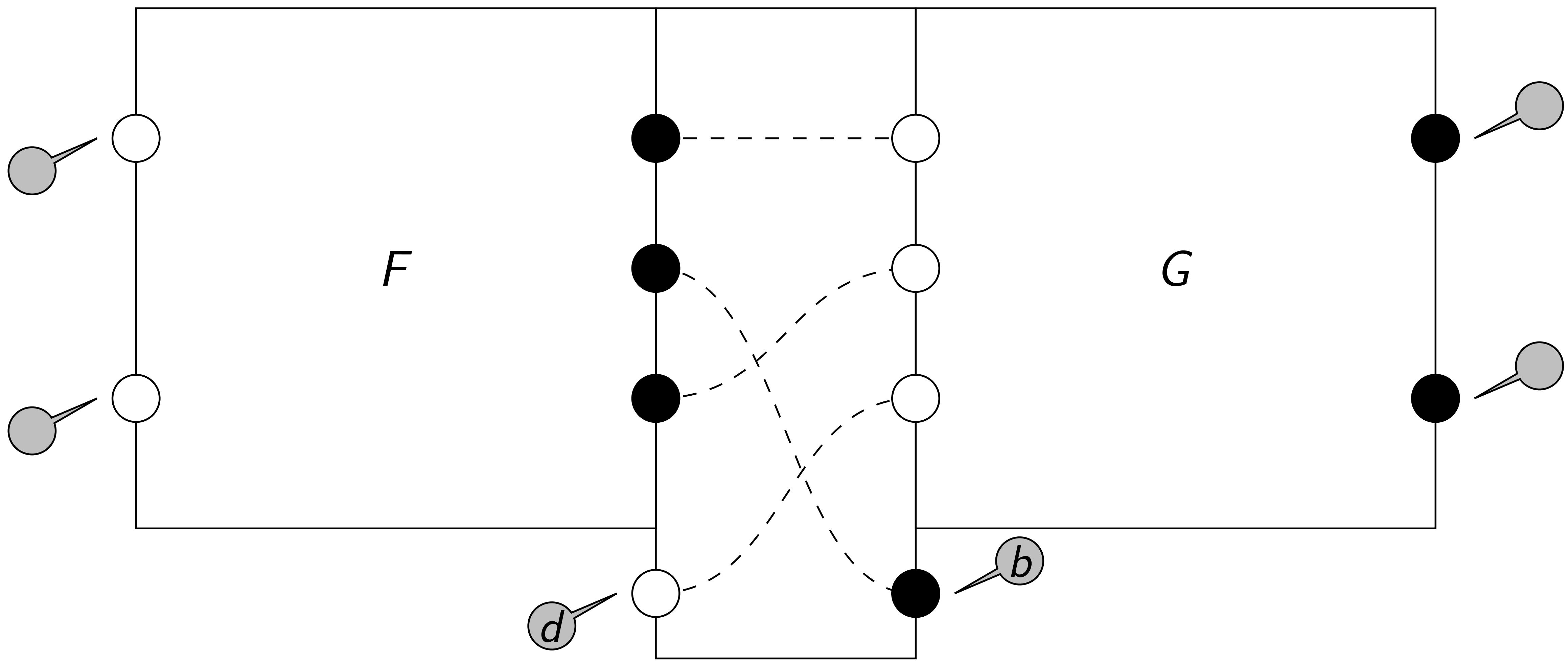
Arbitrary Composition



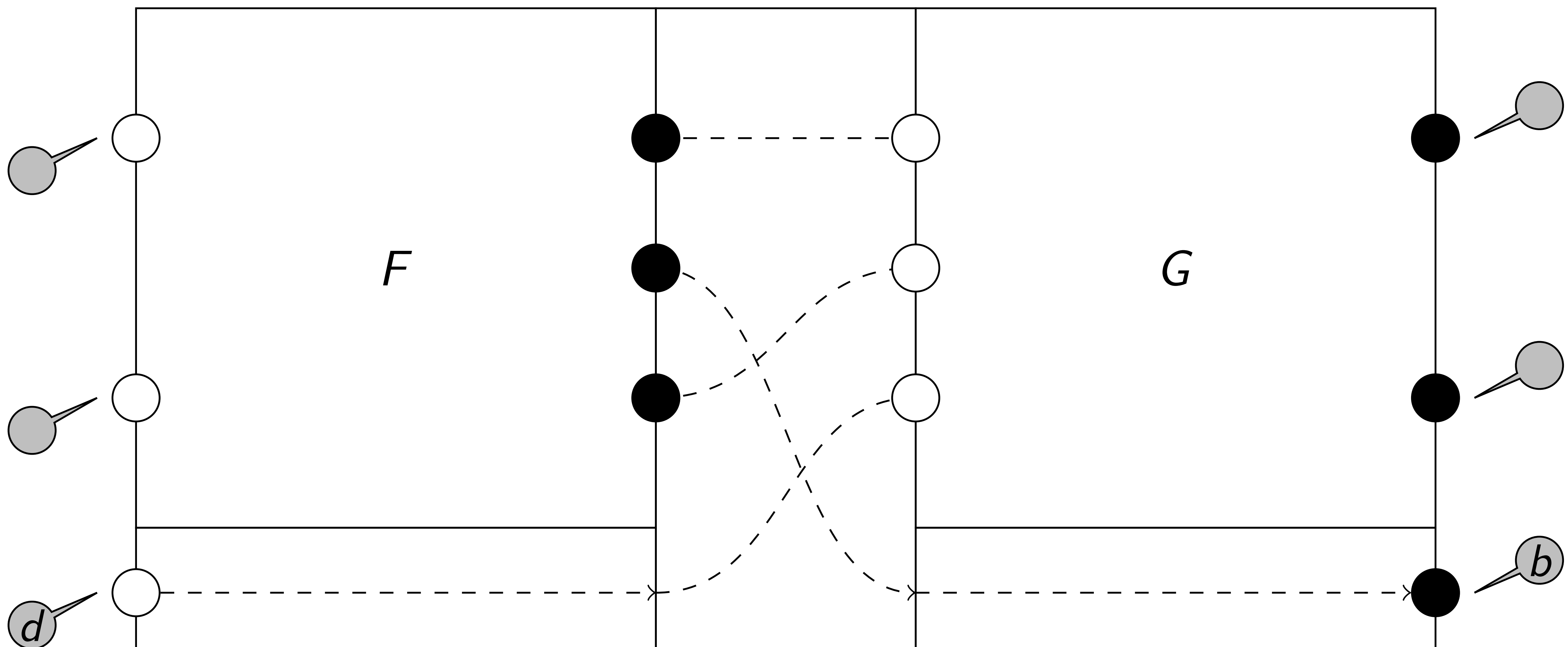
Arbitrary Composition



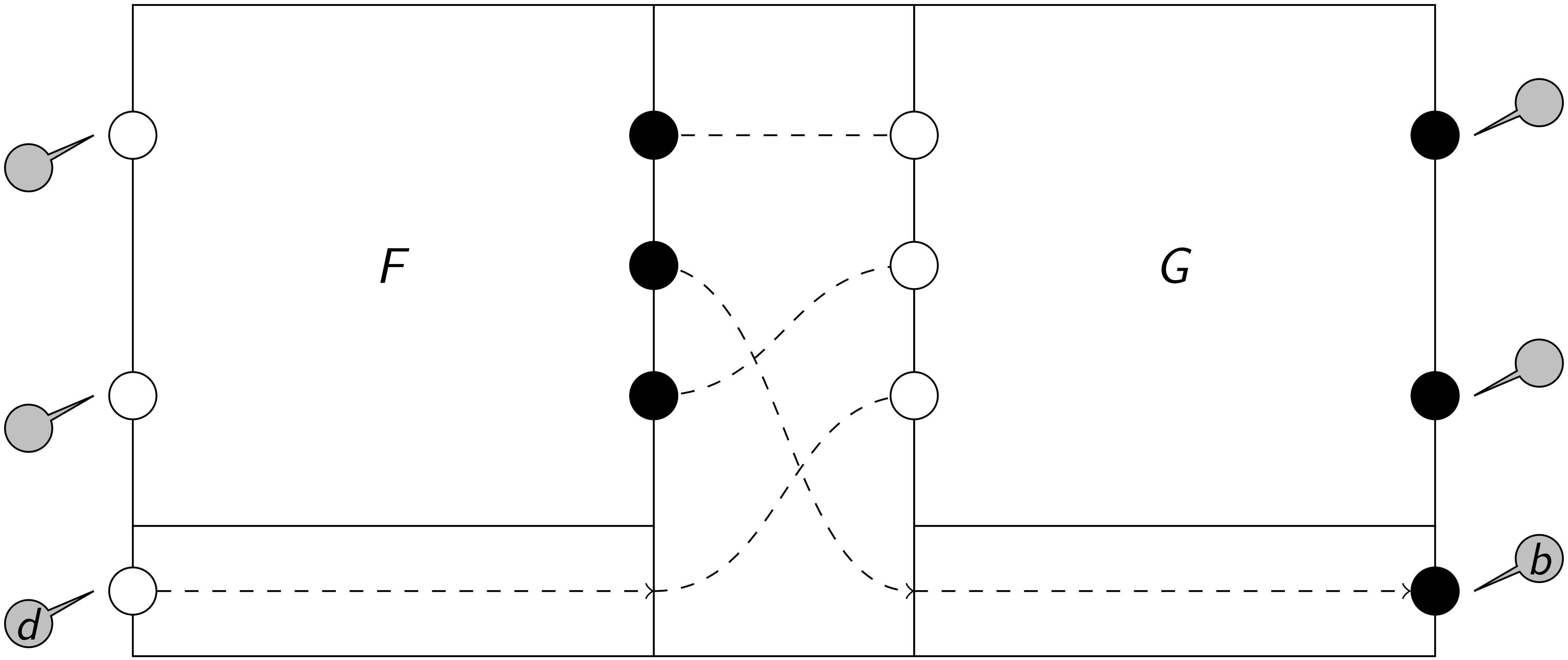
Arbitrary Composition



Arbitrary Composition

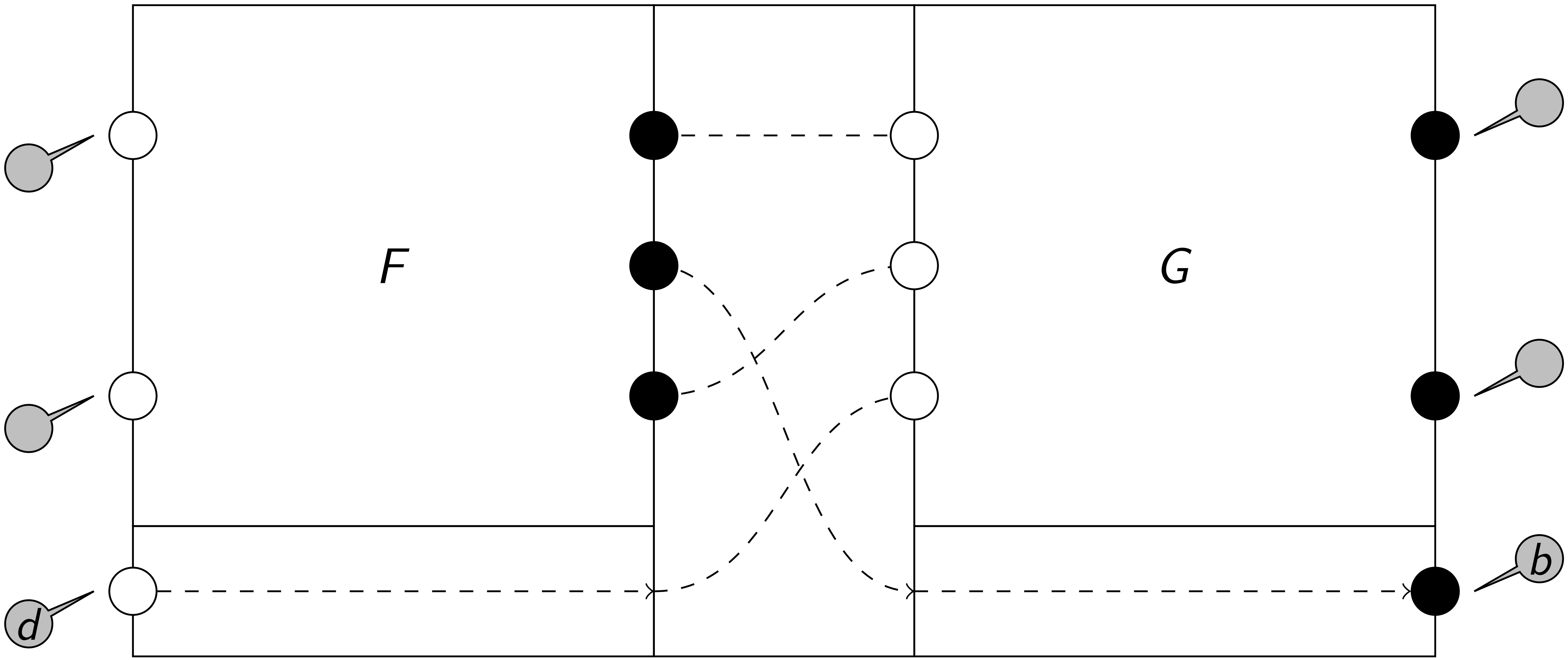


Arbitrary Composition



$$F \circ_L G = (F \otimes Id) \circ \Pi_{(1423)} \circ (G \otimes Id)$$

Arbitrary Composition



$$F \circ_L G = (F \otimes Id) \circ \begin{pmatrix} \delta & & & \\ & & & \delta \\ & \delta & & \\ & & \delta & \end{pmatrix} \circ (G \otimes Id)$$

Case Study

Case Study

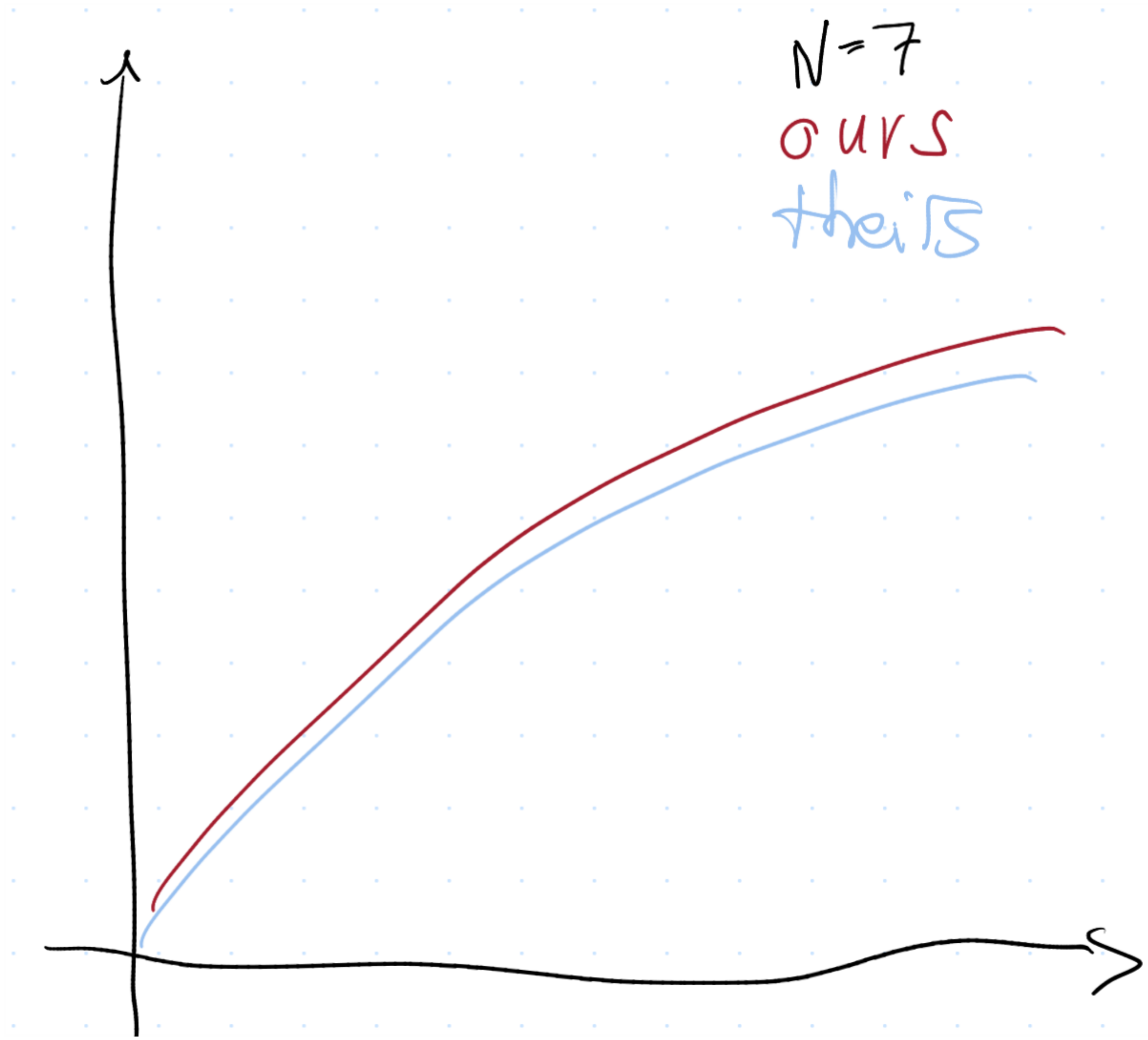


Figure: How incredibly good we are compared to the literature

Overview

- **Composition**: $f \circ g$
  Glueing horizontally
- **Product**: $f \otimes g$
  Glueing vertically
- **Permutation**: Π_σ
  Reordering inputs/outputs

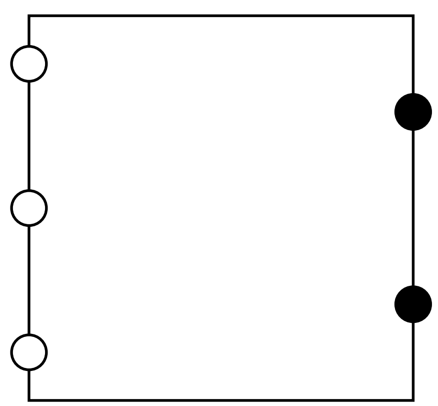
The General Framework

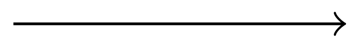
- Have a set of things S with $+$, $*$, 0 and 1

- Composable boxes represented by $m \times n$ matrices

$\rightsquigarrow$ The category **PROP** over the ring $(S, +, *, 0, 1)$

Categorical Notation (String Diagrams) [5]





Categorical Notation (String Diagrams) [5]

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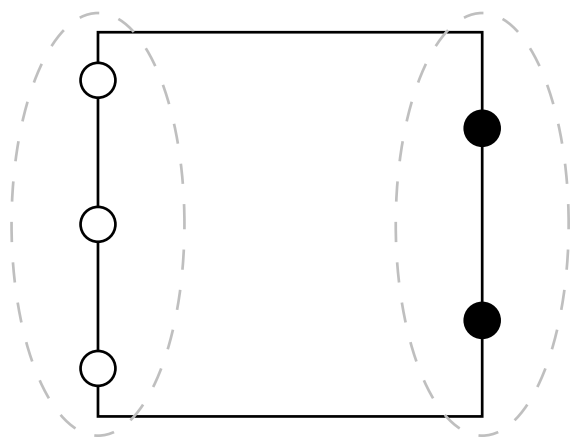
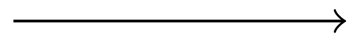
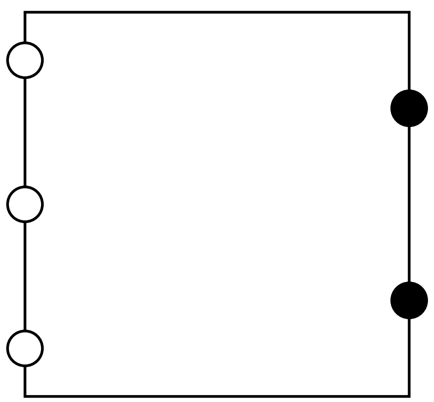


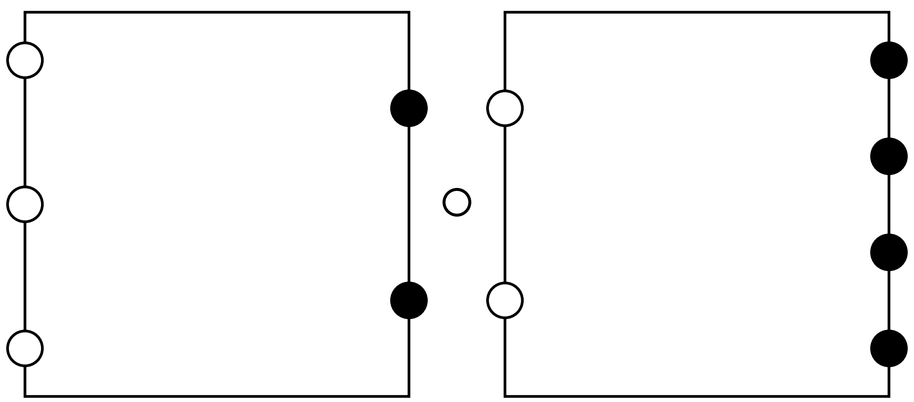


Categorical Notation (String Diagrams) [5]



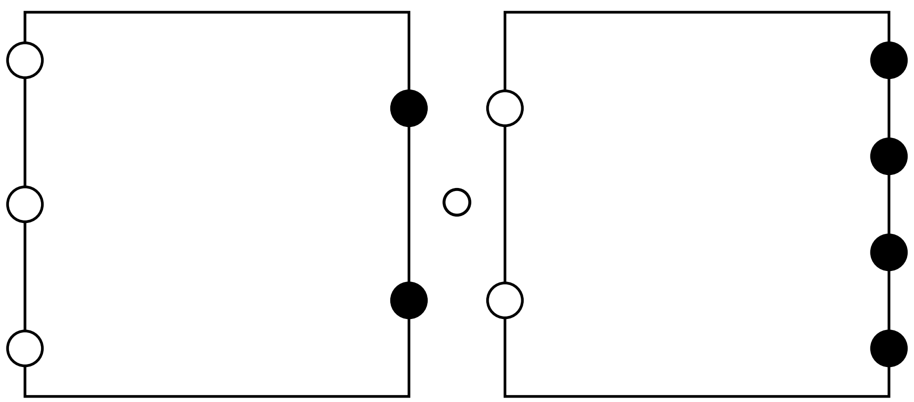
$$3 \longrightarrow 2$$

Categorical Notation (String Diagrams) [5]



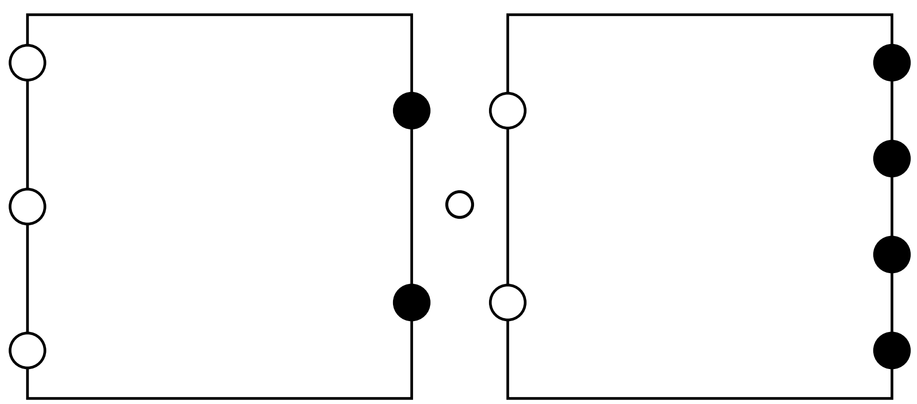
$3 \longrightarrow 2$

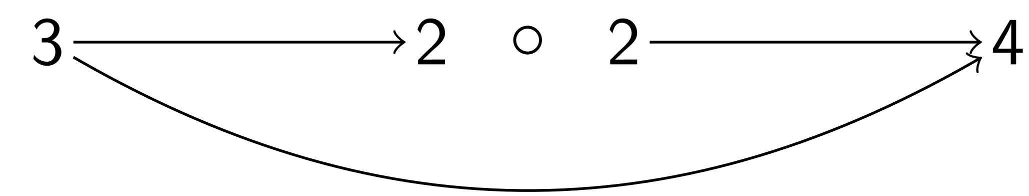
Categorical Notation (String Diagrams) [5]



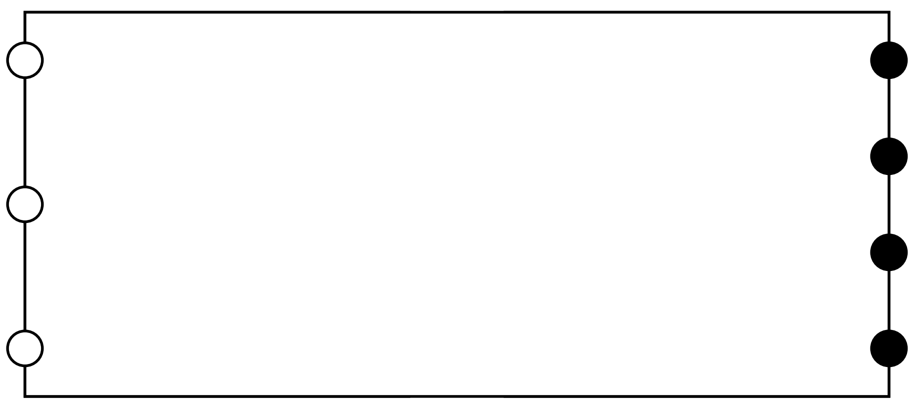
$$3 \longrightarrow 2 \quad \circ \quad 2 \longrightarrow 4$$

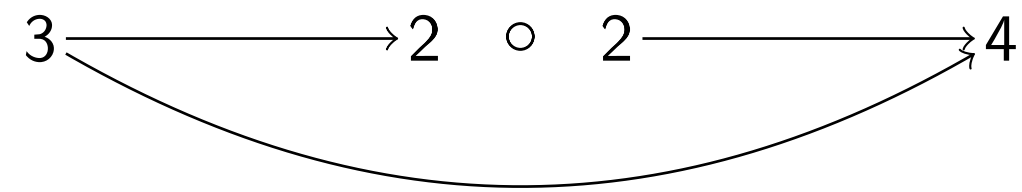
Categorical Notation (String Diagrams) [5]



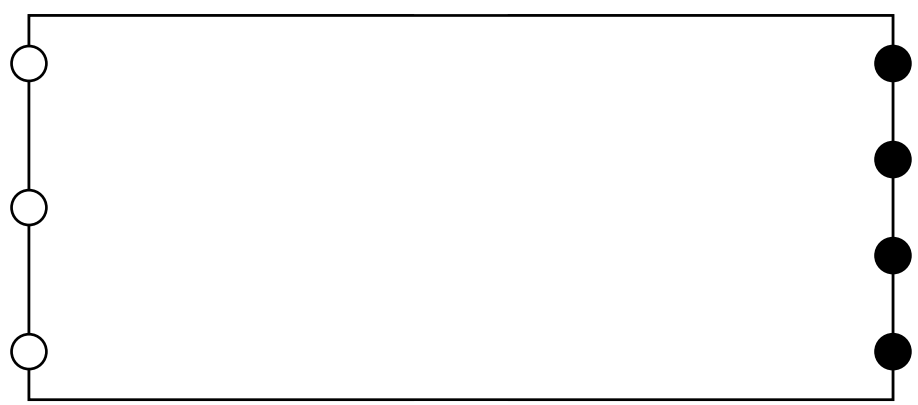


Categorical Notation (String Diagrams) [5]



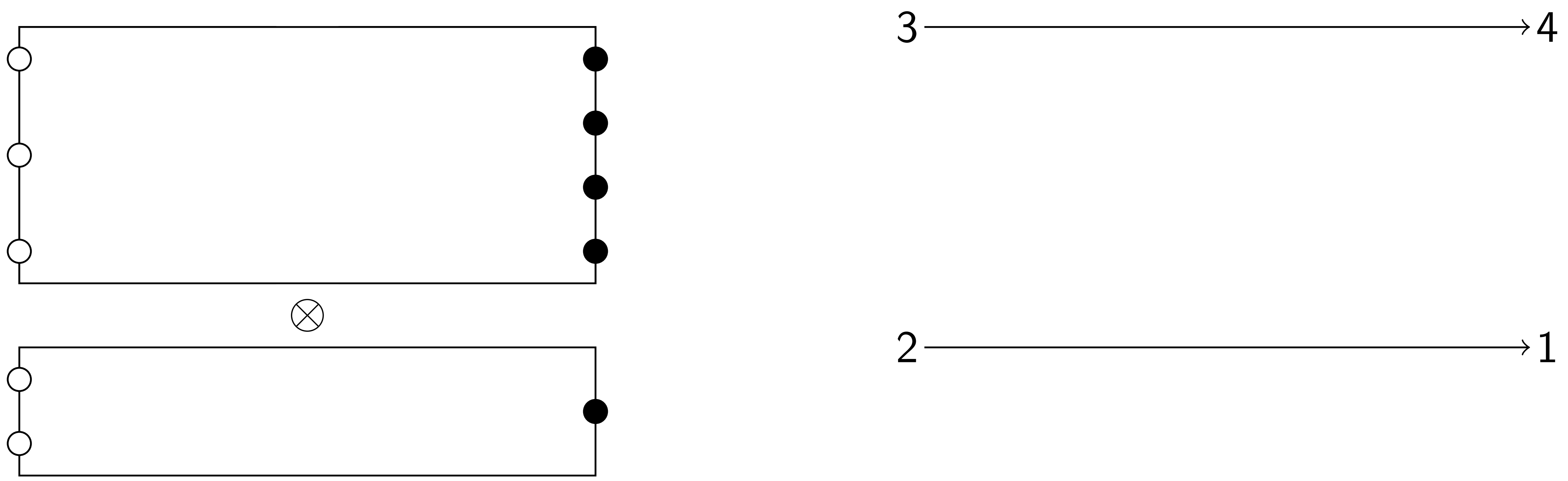


Categorical Notation (String Diagrams) [5]

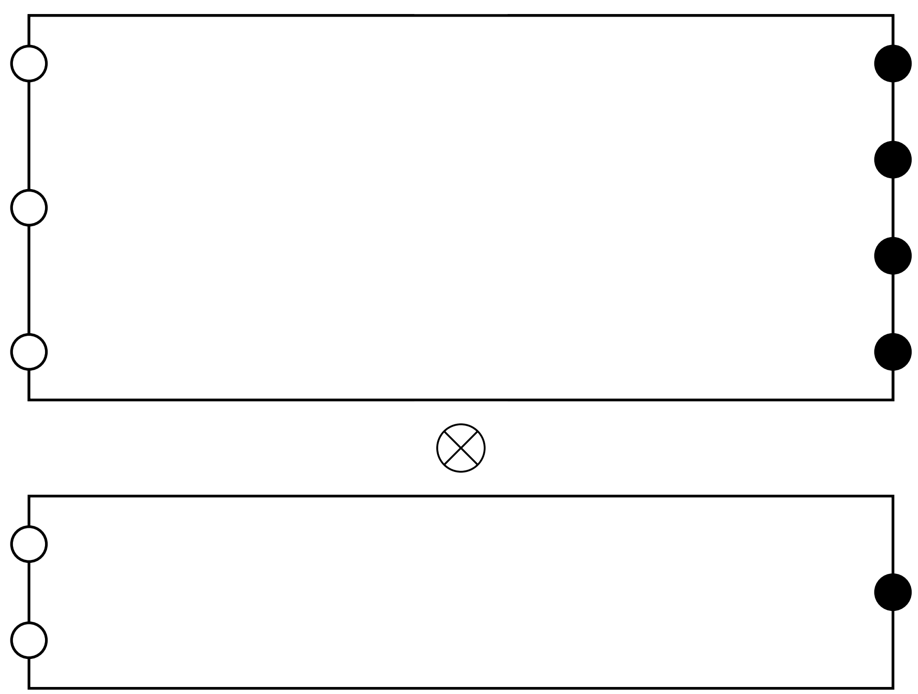


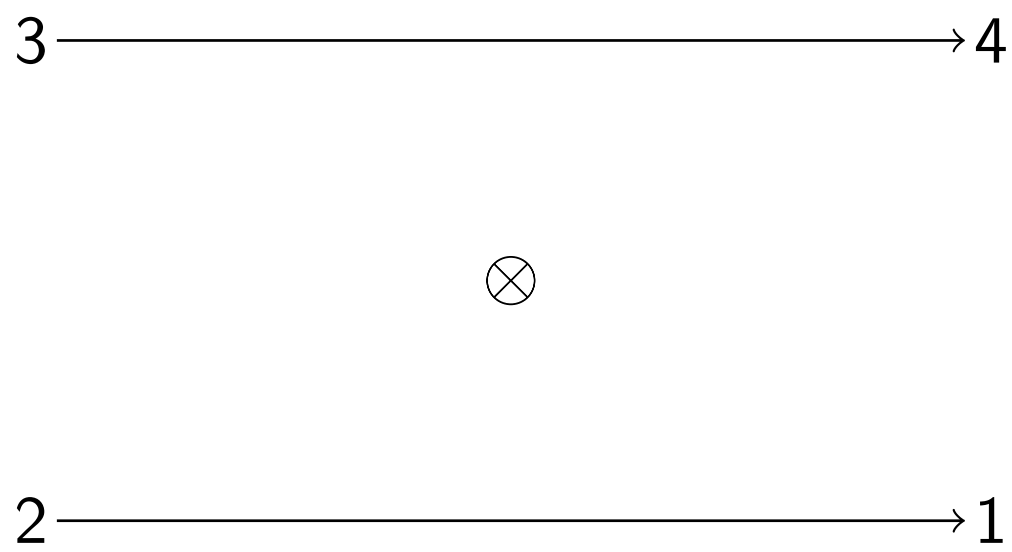
3 $\longrightarrow$ 4

Categorical Notation (String Diagrams) [5]

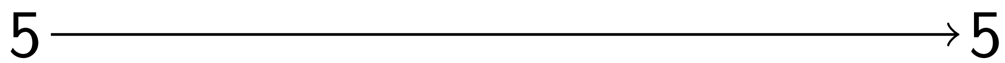


Categorical Notation (String Diagrams) [5]





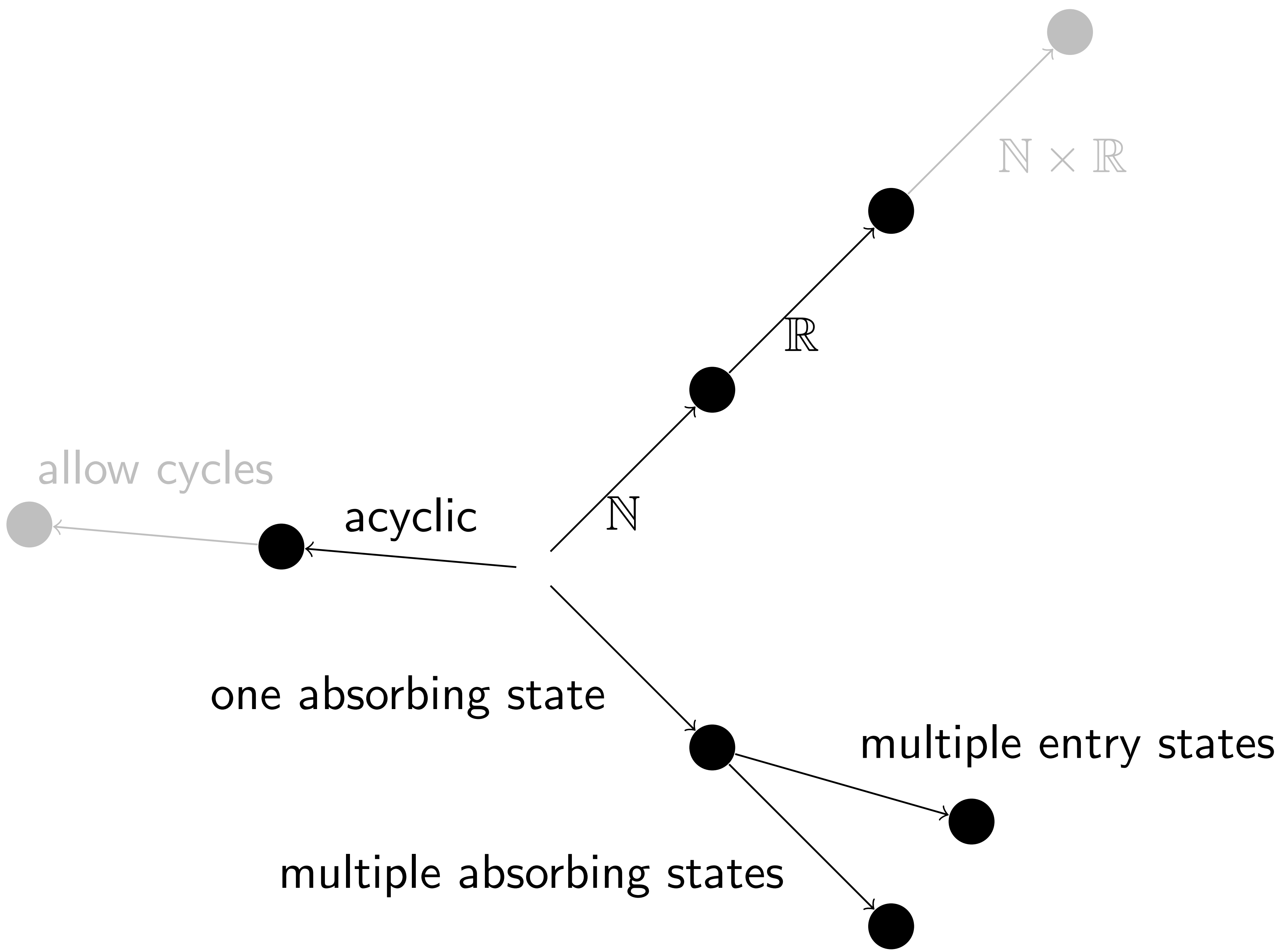
Categorical Notation (String Diagrams) [5]



Take-Away

- Modelling stochastic systems in continuous time?
  - Consider using *APD*s
- Compositional Calculus can be made great!
- Monoidal Categories are not so scary
  - **PROP**s Make composition very easy
  - Heavy-lifting happens under the hood

The future



$\mathbb{N} \times \mathbb{R}$
$\mathbb{R}$
allow cycles
acyclic
$\mathbb{N}$
one absorbing state
multiple entry states
multiple absorbing states

References

[1] Sally McClean et al. “A modeling framework that combines markov models and discrete-event simulation for stroke patient care”. In: *ACM Transactions on Modeling and Computer Simulation (TOMACS)* 21.4 (2011). Publisher: ACM New York, NY, USA, pp. 1–26. DOI: https://dl.acm.org/doi/10.1145/2000494.2000498.

[2] Sally McClean et al. “Using mixed phase-type distributions to model patient pathways”. In: *2010 IEEE 23rd International Symposium on Computer-Based Medical Systems (CBMS)*. 2010, pp. 172–177. DOI: 10.1109/CBMS.2010.6042636.

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